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The transmuted geometric-quadratic hazard rate distribution: development, properties, characterizations and applications

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Abstract

We propose a five parameter transmuted geometric quadratic hazard rate (TG-QHR) distribution derived from mixture of quadratic hazard rate (QHR), geometric and transmuted distributions via the application of transmuted geometric-G (TG-G) family of Afify et al.(Pak J Statist 32(2), 139-160, 2016). Some of its structural properties are studied. Moments, incomplete moments, inequality measures, residual life functions and some other properties are theoretically taken up. The TG-QHR distribution is characterized via different techniques. Estimates of the parameters for TG-QHR distribution are obtained using maximum likelihood method. The simulation studies are performed on the basis of graphical results to illustrate the performance of maximum likelihood estimates (MLEs) of the TG-QHR distribution. The significance and flexibility of TG-QHR distribution is tested through different measures by application to two real data sets.

Keywords: Quadratic hazard rate, Geometric distribution, Characterizations, Maximum likelihood estimation

Introduction

Generalizations and extensions of the probability distributions are more flexible and suitable for many real data sets as compared to the classical distributions. Azzalini (1985) derived Skewed Family with additional skewing parameter. Gupta et al. (1998) developed exponentiated family. Marshall and Olkin (1997) introduced a parameter to the family of distributions. Eugene et al. (2002) established family formed from beta distribution. Jones (2004) also presented a family generated from beta distribution. The transmuted family was presented by Shaw and Buckley (2007). Zografos Balakrishnan (2009) established family based on gamma distribution. Cordeiro and Castro (2011) developed family produced from Kumaraswamy distribution. Alexander et al. (2012) studied family based on McDonald distribution. Cordeiro et al. (2013) studied exponentiated generalized family of distribution. Torabi and Montazari (2014) studied family of distributions created from logistic distribution. Elbatal and Butt (2014) developed Kumaraswamy quadratic hazard rate distribution. Alizadeh et al. (2015a) developed Kumaraswamy Marshal Olkin family. Alizadeh et al. (2015b) also proposed Kumaraswamy odd log-logistic family. Yousof et al.

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(2015) studied transmuted exponentiated generalized-G family of distributions. Afify et al. (2016a) developed a family of distributions called Kumaraswamy transmuted-G family. Afify et al. (2016b) presented transmuted geometric-G family (TG-G). Cordeiro et al. (2016) presented beta odd log-logistic family of distributions. Nofal et al. (2017) studied transmuted geometric Weibull distribution in terms of mathematical properties, characterizations and regression models. Cordeiro et al. (2017) studied generalized odd log-logistics family of distributions in terms of various characteristics and applications. Alizadeh et al., (2018) proposed odd power Cauchy family of distributions and studied its properties and regression models. Yousof et al. (2018) developed a family of distributions on the basis of Burr Hatke differential equation.

The TG-G family (Afify et al.; 2016b) has been developed on the basis of the T-X idea (Alzaatreh et al.; 2013). Let $g(x) = 1 + \lambda - 2\lambda x,\ 0 < x < 1$ and $W(G(x)) = \frac{\theta G(x)}{1-(1-\theta)G(x)}$, where $W(G(x))$ is non-decreasing function of X and $G(x)$ is a base line cumulative distribution function (cdf) of X [see Alzaatreh et al.; 2013 for W(G(x))]. Then, the cdf of TG-G family is given by

$$F(x) = \int_0^{\frac{\theta G(x)}{1-(1-\theta)G(x)}} (1+\lambda-2\lambda z)dz.$$

Another definition of TG-G family has been given by Afify et al. (2016b) as follows;

Let X_1 and X_2 be independent and identically distributed (i.i.d.) random variables from $\frac{\theta G(x)}{1-(1-\theta)G(x)}$. Then, the cdf of TG-G family is

$$F(x) = \frac{\theta G(x)}{1-(1-\theta)G(x)}\left[1+\frac{\lambda G(x)}{1-(1-\theta)G(x)}\right], \quad |\lambda| \leq 1, \theta \in (0,1), x \geq 0.$$

Proof

Consider the following order statistics:

$$X_{1:2} = \min(X_1, X_2) \ and \ X_{2:2} = \max(X_1, X_2)$$

and

let $X \overset{d}{=} X_{1:2}$, with probability $\frac{\lambda+1}{2}$,

$X \overset{d}{=} X_{2:2}$, with probability $\frac{1-\lambda}{2}$,

where $|\lambda| \leq 1$. The cdf of X is

$$F(x) = \left(\frac{\lambda+1}{2}\right)\mathrm{P}(X_{1:2} \leq x) + \left(\frac{1-\lambda}{2}\right)\mathrm{Pr}(X_{2:2} \leq x),$$

or

$$F(x) = \left(\frac{1-\lambda}{2}\right)\left[1-\left[1-\frac{\theta G(x)}{1-(1-\theta)G(x)}\right]^2\right] + \left(\frac{1-\lambda}{2}\right)\left[\frac{\theta G(x)}{1-(1-\theta)G(x)}\right]^2,$$

$$F(\mathrm{x};\theta,\lambda) = \frac{\theta G(x)}{1-(1-\theta)G(x)}\left[1+\frac{\lambda G(x)}{1-(1-\theta)G(x)}\right], \quad |\lambda| \leq 1, \theta \in (0,1), x \geq 0, \tag{1}$$

The probabilty density function (pdf) of TG-G family is

$$f(\mathrm{x};\theta,\lambda)=\frac{\theta g(x)}{[1+(\theta-1)G(x)]^2}\left[1+\lambda-\frac{2\lambda\theta G(x)}{(1+(\theta-1)G(x))}\right],\ \ |\lambda|\le 1,\theta\in(0,1),x>0, \tag{2}$$

where $g(x)$ is the baseline pdf.

The basic motivations for proposing the TG-QHR distribution are: (i) to generate distributions with arc, positively skewed, negatively skewed and symmetrical shaped; (ii) to obtain increasing, decreasing and inverted bathtub hazard rate function; (iii) to serve as the best alternative model for the current models to explore and modeling real data in economics, life testing, reliability, survival analysis manufacturing and other areas of research and (iv) to provide better fits than other sub-models.

Our interest is to study the TG-QHR distribution along with its properties, applications and examine the usefulness of this distribution for modeling phenomena compared to the sub-models.

This article is composed as follows. In Section "TG-QHR distribution", TG-QHR distribution is introduced. In Section "Structural properties of TG-QHR distribution", TG-QHR distribution is studied in terms of various structural properties, plots, sub-models and descriptive measures on the basis of quantiles. In Section "Moments and inequality measures", moments, incomplete moments, residual life functions and inequality measures and some other properties are theoretically derived. In Section "Characterizations", TG-QHR distribution is characterized via (i) ratio of truncated moments; (ii) hazard function; (iii) reverse hazard rate function and (iv) elasticity function. In Section "Maximum likelihood estimation", estimates of the parameters of TG-QHR distribution are obtained via maximum likelihood method. In Section "Simulation studies", simulation studies are performed on the basis of graphical results to illustrate the performance of MLEs. In Section "Applications", the significance and flexibility of TG-QHR distribution is tested through different measures by application to two real data sets. Goodness of fit of TG-QHR distribution is checked via different methods. Conclusion is given in Section "Conclusions".

TG-QHR distribution

The goal of this article is to propose a five parameter transmuted geometric quadratic hazard rate (TG-QHR) distribution from mixture of QHR, geometric and transmuted distributions by the application of TG-G family.

Bain (1974) developed quadratic hazard rate (QHR) distribution from the following quadratic function

$$A(x)=\alpha+\beta x+\gamma x^2,\mathrm{x}>0,\ \ \alpha>0,\ \gamma>0,\ \beta>-2\sqrt{\alpha\gamma}. \tag{3}$$

The cdf of the random variable X with QHR distribution and parameters $\alpha,\beta\ and\ \gamma$ is

$$F_{QHR}(x)=1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)},\alpha>0,\ \gamma>0,\ \beta>-2\sqrt{\alpha\gamma},x\ge 0, \tag{4}$$

where $Q(x|\alpha,\beta,\gamma)=\alpha x+\frac{\beta}{2}x^2+\frac{\gamma}{3}x^3$.

The pdf of the random variable X with QHR distribution and parameters $\alpha,\beta\ and\ \gamma$ is

$$f_{QHR}(x) = A(x|\alpha,\beta,\gamma)\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}, x > 0. \tag{5}$$

The pdf and cdf of a random variable X with TG-QHR distribution are obtained by inducting (4) and (5) in (1) and (2) as follows

$$f(x) = \frac{\theta\, A(x|\alpha,\beta,\gamma)\ \ \mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}}{\left[1+(\theta-1)\left(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\right)\right]^2}\left[1+\lambda-\frac{2\lambda\theta\left(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\right)}{\left(1+(\theta-1)\left(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\right)\right)}\right], x > 0, \tag{6}$$

$$F(x) = \frac{\theta\left(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\right)}{\left[1+(\theta-1)\left(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\right)\right]}\left[1+\frac{\lambda\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}}{\left(1+(\theta-1)\left(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\right)\right)}\right],\ \ x \geq 0, \tag{7}$$

where $\alpha > 0, \quad \beta > -2\sqrt{\alpha\gamma}, \gamma > 0, |\lambda| \leq 1\ and\ \ \theta \in (0,1)$ are parameters.

Useful expansions and mixture representation

The density function for TG-QHR can be written as

$$f(x) = \frac{\theta\, A(x|\alpha,\beta,\gamma)\ \ \mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}}{\left[1+(\theta-1)\left(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\right)\right]^2}\left[1+\lambda-\frac{2\lambda\theta\left(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\right)}{\left(1+(\theta-1)\left(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\right)\right)}\right], x > 0,$$

$$(1+z)^{-a} = \sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(a+k)}{\Gamma(a)}\frac{z^k}{k!}$$

$$f(x) = \left[(1+\lambda)\theta\sum_{k=0}^{\infty}\frac{\Gamma(2+k)}{\Gamma(2)}\frac{(\theta-1)^k}{k!}\sum_{j=0}^{k}(-1)^{j+k}\binom{k}{j}-2\lambda\theta^2\sum_{k=0}^{\infty}\frac{\Gamma(3+k)}{\Gamma(3)}\frac{(\theta-1)^k}{k!}\sum_{j=0}^{k+1}(-1)^{j+k}\binom{k+1}{j}\right]$$
$$A(x|\alpha,\beta,\gamma)\mathrm{e}^{-(j+1)Q(x|\alpha,\beta,\gamma)}.$$

The pdf of TG-QHR distribution can be written as the mixture of exp-G densities.

Structural properties of TG-QHR distribution

The survival, hazard, reverse hazard, cumulative hazard functions and Mills ratio of a random variable X with TG-QHR distribution are given, respectively, by

$$S(x) = 1-\frac{\theta\left(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\right)}{\left[1+(\theta-1)\left(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\right)\right]}\left[1+\frac{\lambda\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}}{\left(1+(\theta-1)\left(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\right)\right)}\right], \tag{8}$$

$$h(x) = A(x|\alpha,\beta,\gamma)\ \mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\left[\frac{1}{\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}}+\frac{\lambda}{1-\lambda\left(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\right)}\right], \tag{9}$$

$$r(x) = A(x|\alpha,\beta,\gamma)\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\left[\frac{1}{\left(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\right)}-\frac{\lambda}{1+\lambda\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}}\right], \tag{10}$$

$$H(x) = -\ \ln\left\{1-\frac{\theta\left(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\right)}{\left[1+(\theta-1)\left(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\right)\right]}\left[1+\frac{\lambda\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}}{\left(1+(\theta-1)\left(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\right)\right)}\right]\right\}, \tag{11}$$

and

$$m(x) = \mathrm{e}^{Q(x|\alpha,\beta,\gamma)}\left[\frac{A(x|\alpha,\beta,\gamma)}{\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}}+\frac{A(x|\alpha,\beta,\gamma)\lambda}{1-\lambda\left(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\right)}\right]^{-1}. \tag{12}$$

The generalized hazard $g_1(x) = -\frac{d\ \ln S(x)}{d\ \ln x} = xh(x)$ of TG-QHR distribution is

$$g_1(x) = xA(x|\alpha,\beta,\gamma)\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\left[\frac{1}{\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}} + \frac{\lambda}{1-\lambda\left(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\right)}\right]. \tag{13}$$

The elasticity $e(x) = \frac{d\ln \mathrm{F}(x)}{d\ \ln x} = xr(x)$of TG-QHR distribution is

$$e(x) = xA(x|\alpha,\beta,\gamma)\,\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\left[\frac{1}{\left(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\right)} - \frac{\lambda}{1+\lambda \mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}}\right]. \tag{14}$$

Shapes of the TG-QHR density and hazard rate functions

The TG-QHR density is arc, positively skewed, positively skewed and symmetrical distribution (Fig. 1a). The TG-QHR hazard is increasing, decreasing and inverted bathtub hazard rate function (Fig. 1b).

Descriptive measures based on quantiles

In this sub-section, descriptive measures on the basis of quantiles are taken up.

The quantile function of TG-QHR distribution is

$$\alpha x_q + \frac{\beta}{2}{x_q}^2 + \frac{\gamma}{3}{x_q}^3 + \ \ln\left[1-\left[\frac{\lambda+1-\sqrt{(\lambda+1)^2-4\lambda q}}{2\lambda\theta+(1-\theta)\left[\lambda+1-\sqrt{(\lambda+1)^2-4\lambda q}\right]}\right]\right] = 0. \tag{15}$$

Median of TG-QHR distribution is

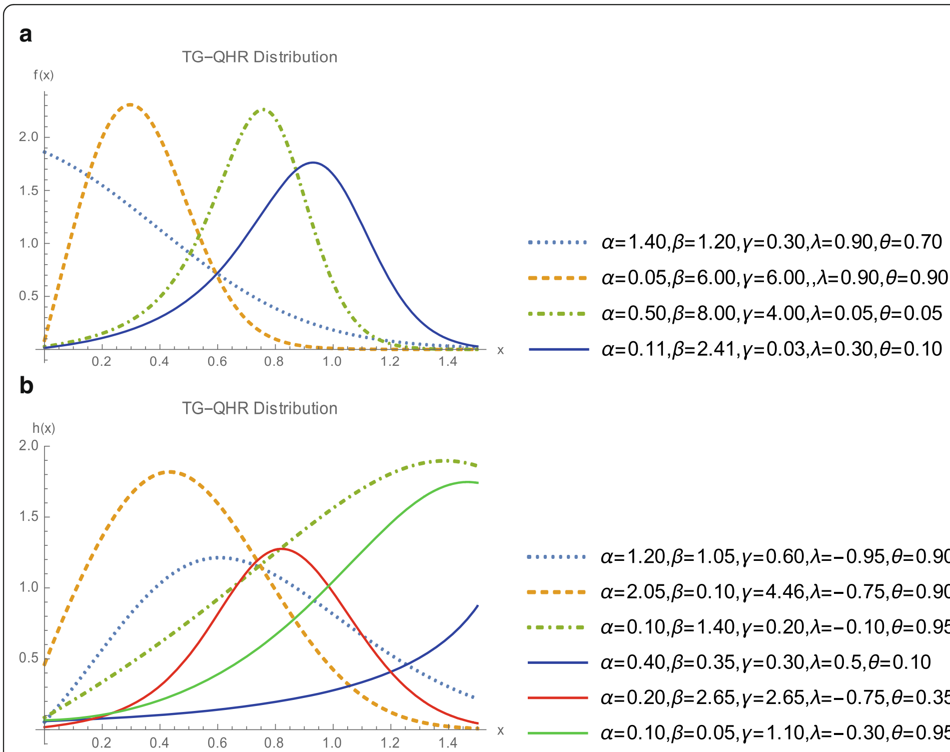

Fig. 1 **a** Plots of pdf of TG-QHR Distribution. **b** Plots of hrf of TG-QHR Distribution

$$\alpha x_{Med.} + \frac{\beta}{2} x_{Med.}{}^{2} + \frac{\gamma}{3} x_{Med.}{}^{3} + \ln\left[1 - \left[\frac{\lambda + 1 - \sqrt{(\lambda+1)^2 - 2\lambda}}{2\lambda\theta + (1-\theta)\left[\lambda + 1 - \sqrt{(\lambda+1)^2 - 2\lambda}\right]}\right]\right] = 0. \tag{16}$$

The random number generator of TG-QHR distribution is

$$\alpha X + \frac{\beta}{2} X^2 + \frac{\gamma}{3} X^3 + \ln\left[1 - \left[\frac{\lambda + 1 - \sqrt{(\lambda+1)^2 - 4\lambda Z}}{2\lambda\theta + (1-\theta)\left[\lambda + 1 - \sqrt{(\lambda+1)^2 - 4\lambda Z}\right]}\right]\right] = 0, \tag{17}$$

where the random variable Z has the uniform distribution on (0, 1).

Sub models of TG-QHR distribution

The TG-QHR has wide applications in life testing, survival analysis and reliability theory. The TG-QHR has the following sub models (Table 1).

Moments and inequality measures

In this section, moments about the origin, incomplete moments, inequality measures, residual life functions and some other properties are theoretically derived.

Moments about the origin

The r^{th} moment about the origin of the random variable X with TG-QHR distribution is

Table 1 Sub-models of TG-QHR distribution

Sr.No.	θ	λ	α	β	γ	Name of Distribution
1	θ	λ	α	β	γ	Transmuted Geometric Quadratic Hazard Rate(TG-QHR)
2	1	λ	α	β	γ	Transmuted Quadratic Hazard Rate(T-QHR)
3	1	0	α	β	γ	Quadratic Hazard Rate(QHR)
4	θ	0	α	β	γ	Quadratic Hazard Rate- Geometric (QHR-G) [Okasha et al.; 2015]
5	θ	λ	0	0	γ	Transmuted Geometric Weibull(TG-W) (Nofal et al.; 2017)
6	θ	λ	0	β	0	Transmuted Geometric Rayleigh(TG-R)
7	θ	λ	α	0	0	Transmuted Geometric Exponential (TG-E)
8	θ	λ	α	β	0	Transmuted Geometric Linear failure rate (TG-LFR)
9	1	λ	0	0	γ	Transmuted Weibull (T-W) [Khan et al.; 2017]
10	1	λ	0	β	0	Transmuted Rayleigh(T-R)[Merovci, F.; 2013]
11	1	λ	α	0	0	Transmuted Exponential (T-E)
12	1	λ	α	β	0	Transmuted Linear failure rate (T-LFR)
13	θ	0	0	0	γ	Weibull Geometric (W-G)[Barreto-Souza et al.; 2011]
14	θ	0	0	β	0	Rayleigh Geometric(R-G)
15	θ	0	α	0	0	Exponential Geometric(E-E)
16	θ	0	α	β	0	Linear failure rate Geometric(LFR-G)
17	1	0	0	0	γ	Weibull (Weibull; 1951)
18	1	0	0	β	0	Rayleigh
19	1	0	α	0	0	Exponential
20	1	0	α	β	0	Linear failure rate(LFR)

$$\mu_r^{/} = E(X^r) = \int_0^\infty x^r f(x)dx,$$

$$\mu_r^{/} = \int_0^\infty x^r \frac{\theta A(x|\alpha,\beta,\gamma)e^{-Q(x|\alpha,\beta,\gamma)}}{\left[1+(\theta-1)(1-e^{-Q(x|\alpha,\beta,\gamma)})\right]^2}\left[1+\lambda-\frac{2\lambda\theta\left(1-e^{-Q(x|\alpha,\beta,\gamma)}\right)}{(1+(\theta-1)(1-e^{-Q(x|\alpha,\beta,\gamma)}))}\right]dx,$$

$$\mu_r^{/} = E(X^r) = \int_0^\infty x^r \left[\begin{array}{l}(1+\lambda)\theta\sum_{k=0}^{\infty}\frac{\Gamma(2+k)}{\Gamma(2+k)}\frac{(\theta-1)^k}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\left(\alpha+\beta x+\gamma x^2\right)e^{-(j+1)Q(x|\alpha,\beta,\gamma)}- \\ \qquad 2\lambda\theta^2\sum_{k=0}^{\infty}\frac{\Gamma(3+k)}{\Gamma(3+k)}\frac{(\theta-1)^k}{k!}\sum_{j=0}^{k+1}(-1)^j\binom{k+1}{j}\left(\alpha+\beta x+\gamma x^2\right)e^{-(j+1)Q(x|\alpha,\beta,\gamma)}\end{array}\right]dx,$$

$$\mu_r^{/} = E(X^r) = \int_0^\infty \left[\begin{array}{l}(1+\lambda)\theta\sum_{j=0}^{k}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{\Gamma(2+k)}{\Gamma(2+k)}\frac{(\theta-1)^k}{k!l!\mathrm{m}!}(-1)^{j+l+m}\binom{k}{j}(j+1)^{l+m} \\ \qquad \times\left(\frac{\beta}{2}\right)^l\left(\frac{\gamma}{3}\right)^m\left(\alpha x^{2l+3m+r}+\beta x^{2l+3m+r+1}+\gamma x^{2l+3m+r+2}\right)e^{-\alpha(j+1)x} \\ -2\lambda\theta^2\sum_{j=0}^{k+1}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{\Gamma(3+k)}{\Gamma(3+k)}\frac{(\theta-1)^k}{k!l!\mathrm{m}!}(-1)^{j+l+m}\binom{k+1}{j}(j+1)^{l+m}\left(\frac{\beta}{2}\right)^l\left(\frac{\gamma}{3}\right)^m \\ \qquad \times\left(\alpha x^{2l+3m+r}+\beta x^{2l+3m+r+1}+\gamma x^{2l+3m+r+2}\right)e^{-\alpha(j+1)x}\end{array}\right]dx,$$

$$\mu_r^{/} = E(X^r) = \left[\begin{array}{l}(1+\lambda)\theta\sum_{j=0}^{k}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{\Gamma(2+k)}{\Gamma(2+k)}\frac{(\theta-1)^k}{k!l!\mathrm{m}!}(-1)^{j+l+m}\binom{k}{j}(j+1)^{l+m} \\ \qquad \times\left(\frac{\beta}{2}\right)^l\left(\frac{\gamma}{3}\right)^m\frac{\alpha\Gamma(r+2l+3m+1)}{[\alpha(j+1)]^{(r+2l+3m+1)}}+\frac{\beta\Gamma(r+2l+3m+2)}{[\beta(j+1)]^{(r+2l+3m+2)}}+\frac{\gamma\Gamma(r+2l+3m+3)}{[\gamma(j+1)]^{(r+2l+3m+3)}} \\ -2\lambda\theta^2\sum_{j=0}^{k+1}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{\Gamma(3+k)}{\Gamma(3+k)}\frac{(\theta-1)^k}{k!l!\mathrm{m}!}(-1)^{j+l+m}\binom{k+1}{j}(j+1)^{l+m} \\ \qquad \times\left(\frac{\beta}{2}\right)^l\left(\frac{\gamma}{3}\right)^m\frac{\alpha\Gamma(r+2l+3m+1)}{[\alpha(j+1)]^{(r+2l+3m+1)}}+\frac{\beta\Gamma(r+2l+3m+2)}{[\beta(j+1)]^{(r+2l+3m+2)}}+\frac{\gamma\Gamma(r+2l+3m+3)}{[\gamma(j+1)]^{(r+2l+3m+3)}}\end{array}\right],$$

where $W_{j,k,l,m} = \sum_{j=0}^{k}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{\Gamma(2+k)}{\Gamma(2+k)}\frac{(\theta-1)^k}{k!l!\mathrm{m}!}(-1)^{j+l+m}\binom{k}{j}(j+1)^{l+m}\left(\frac{\beta}{2}\right)^l\left(\frac{\gamma}{3}\right)^m$ and

$$W_{j,k+1,l,m} = \sum_{j=0}^{k+1}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{\Gamma(3+k)}{\Gamma(3+k)}\frac{(\theta-1)^k}{k!l!\mathrm{m}!}(-1)^{j+l+m}\binom{k+1}{j}(j+1)^{l+m}\left(\frac{\beta}{2}\right)^l\left(\frac{\gamma}{3}\right)^m.$$

$$\mu_r^{/} = E(X^r) = \left[\begin{array}{l}(1+\lambda)\theta W_{j,k,l,m}\frac{\alpha\Gamma(r+2l+3m+1)}{[\alpha(j+1)]^{(r+2l+3m+1)}}+\frac{\beta\Gamma(r+2l+3m+2)}{[\beta(j+1)]^{(r+2l+3m+2)}} \\ \quad +\frac{\gamma\Gamma(r+2l+3m+3)}{[\gamma(j+1)]^{(r+2l+3m+3)}}-2\lambda\theta^2W_{j,k+1,l,m}\frac{\alpha\Gamma(r+2l+3m+1)}{[\alpha(j+1)]^{(r+2l+3m+1)}} \\ \quad +\frac{\beta\Gamma(r+2l+3m+2)}{[\beta(j+1)]^{(r+2l+3m+2)}}+\frac{\gamma\Gamma(r+2l+3m+3)}{[\gamma(j+1)]^{(r+2l+3m+3)}}\end{array}\right],$$

$$\mu_r^{/} = E(X^r) = \left[(1+\lambda)\theta W_{j,k,l,m}-2\lambda\theta^2W_{j,k+1,l,m}\right]\frac{\alpha\Gamma(r+2l+3m+1)}{[\alpha(j+1)]^{(r+2l+3m+1)}} + \frac{\beta\Gamma(r+2l+3m+2)}{[\beta(j+1)]^{(r+2l+3m+2)}}+\frac{\gamma\Gamma(r+2l+3m+3)}{[\gamma(j+1)]^{(r+2l+3m+3)}}. \tag{18}$$

Mean and Variance of TG-QHR distribution

$$E(X)=\left((1+\lambda)\theta W_{j,k,l,m}-2\lambda\theta^2 W_{j,k+1,l,m}\right)\left[\frac{\alpha\Gamma(2l+3m+2)}{[\alpha(j+1)]^{(2l+3m+2)}}+\frac{\beta\Gamma(2l+3m+3)}{[\beta(j+1)]^{(2l+3m+3)}}+\frac{\gamma\Gamma(2l+3m+4)}{[\gamma(j+1)]^{(2l+3m+4)}}\right],$$

$$Var(X)=\begin{cases}\left((1+\lambda)\theta W_{j,k,l,m}-2\lambda\theta^2 W_{j,k+1,l,m}\right)\left[\frac{\alpha\Gamma(2l+3m+3)}{[\alpha(j+1)]^{(2l+3m+3)}}+\frac{\beta\Gamma(2l+3m+4)}{[\beta(j+1)]^{(2l+3m+4)}}+\frac{\gamma\Gamma(2l+3m+5)}{[\gamma(j+1)]^{(2l+3m+5)}}\right]\\ -\left((1+\lambda)\theta W_{j,k,l,m}-2\lambda\theta^2 W_{j,k+1,l,m}\right)^2\left[\frac{\alpha\Gamma(2l+3m+2)}{[\alpha(j+1)]^{(2l+3m+2)}}+\frac{\beta\Gamma(2l+3m+3)}{[\beta(j+1)]^{(2l+3m+3)}}+\frac{\gamma\Gamma(2l+3m+4)}{[\gamma(j+1)]^{(2l+3m+4)}}\right]^2.\end{cases}$$

The fractional positive moments about the origin of the random variable X with TG-IW distribution are

$$\mu'_{\frac{r}{s}}=E\left(X^{\frac{r}{s}}\right)=\left[(1+\lambda)\theta W_{j,k,l,m}-2\lambda\theta^2 W_{j,k+1,l,m}\right]\frac{\alpha\,\Gamma\left(\frac{r}{s}+2l+3m+1\right)}{[\alpha(j+1)]^{\left(\frac{r}{s}+2l+3m+1\right)}}+\frac{\beta\,\Gamma\left(\frac{r}{s}+2l+3m+2\right)}{[\beta(j+1)]^{\left(\frac{r}{s}+2l+3m+2\right)}}+\frac{\gamma\,\Gamma\left(\frac{r}{s}+2l+3m+3\right)}{[\gamma(j+1)]^{\left(\frac{r}{s}+2l+3m+3\right)}}. \quad (19)$$

The factorial moments of X with TG- QHR distribution are $E[X]_m=\sum_{r=1}^{m}\phi_r E(X^r)$,

$$E[X]_m=\sum_{r=1}^{m}\phi_r\left((1+\lambda)\theta W_{j,k,l,m}-2\lambda\theta^2 W_{j,k+1,l,m}\right)\left[\frac{\alpha\Gamma(r+2l+3m+1)}{[\alpha(j+1)]^{(r+2l+3m+1)}}+\frac{\beta\Gamma(r+2l+3m+2)}{[\beta(j+1)]^{(r+2l+3m+2)}}+\frac{\gamma\,\Gamma(r+2l+3m+3)}{[\gamma(j+1)]^{(r+2l+3m+3)}}\right], \quad (20)$$

where $[X]_i=X(X+1)(X+2)\ldots(X+i-1)$and ϕ_r is Stirling number of the first kind.

The Mellin transform helps to determine the moments for a probability distribution. The Mellin transform of X with the TG-QHR distribution is $\mathrm{M}\{f(x);s\}=f^*(s)=\int_0^\infty f(x)$ $x^{s-1}dx$.

$$\mathrm{M}\{f(x);s\} = \int_0^\infty x^{s-1} \frac{\theta A(x|\alpha,\beta,\gamma)\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}}{\left[1+(\theta-1)\left(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\right)\right]^2}\left[1+\lambda-\frac{2\lambda\theta\left(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\right)}{\left(1+(\theta-1)\left(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\right)\right)}\right]dx,$$

$$\mathrm{M}\{f(x);s\} = \left[(1+\lambda)\theta W_{j,k,l,m}-2\lambda\theta^2 W_{j,k+1,l,m}\right]\frac{\alpha\Gamma(s+2l+3m)}{[\alpha(j+1)]^{(s+2l+3m)}} + \frac{\beta\Gamma(s+2l+3m+1)}{[\beta(j+1)]^{(s+2l+3m+1)}}+\frac{\gamma\Gamma(s+2l+3m+2)}{[\gamma(j+1)]^{(s+2l+3m+2)}}. \tag{21}$$

The qth central moments, Pearson's measure of skewness and Kurtosis and cumulants of X with TG-QHR distribution are determined from the relationships.

$$\mu_q = \sum_{v=1}^{q}\binom{q}{v}(-1)^v\mu'_v\mu'_{v-q},\quad \gamma_1 = \frac{\mu_3}{(\mu_2)^{\frac{3}{2}}},\quad \beta_2 = \frac{\mu_4}{(\mu_2)^2}\quad \text{and}\quad k_r = \mu'_r-\sum_{c=1}^{r-1}\binom{r-1}{c-1}k_c\,\mu'_{r-c}.$$

The graphical displays to describe the parameter that controls skewness and kurtosis measures of the TG-QHR distribution are added (Fig. 2).

The moment generating function of the random variable X with TG-QHR distribution is

$$M_x(t) = E[e^{tx}] = \int_0^\infty e^{tx}f(x)dx = \sum_{r=0}^{r}\frac{t^r}{r!}\mu'_r,$$

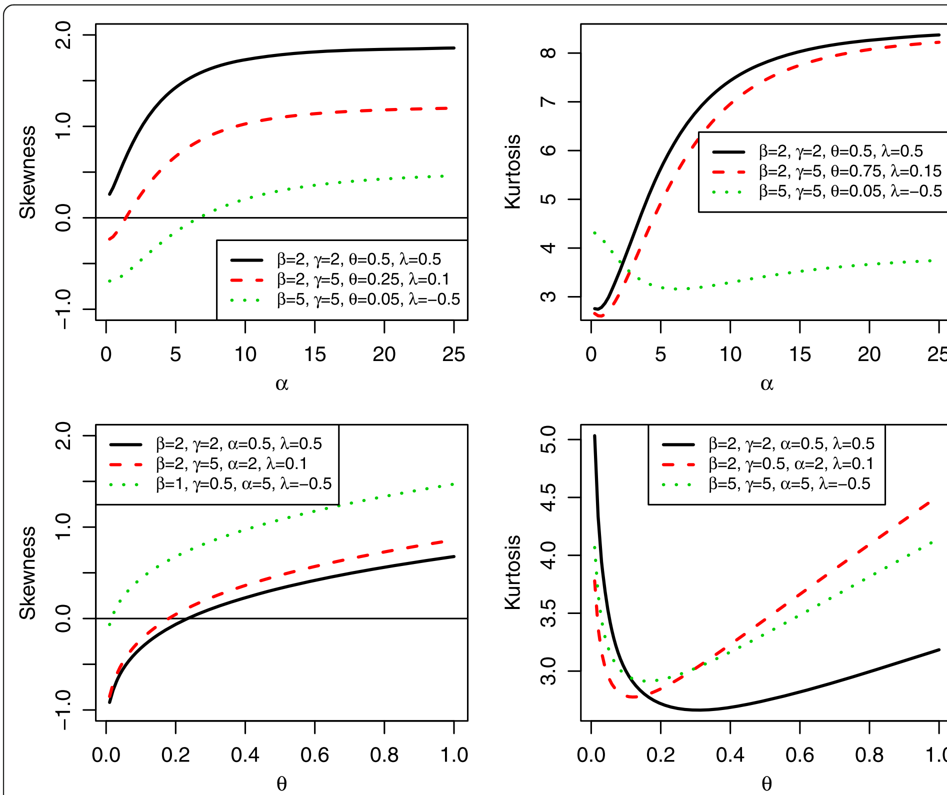

Fig. 2 skewness and kurtosis measures of the TG-QHR distribution

$$M_x(t) = \sum_{r=0}^{r} \frac{t^r}{r!}\left((1+\lambda)\theta W_{j,k,l,m} - 2\lambda\theta^2 W_{j,k+1,l,m}\right)\left[\frac{\alpha\Gamma(r+2l+3m+1)}{[\alpha(j+1)]^{(r+2l+3m+1)}} + \frac{\beta\Gamma(r+2l+3m+2)}{[\beta(j+1)]^{(r+2l+3m+2)}} + \frac{\gamma\Gamma(r+2l+3m+3)}{[\gamma(j+1)]^{(r+2l+3m+3)}}\right]. \tag{22}$$

Incomplete moments

Incomplete moments are used in mean inactivity life, mean residual life function and other inequality measures. The r$^{\text{th}}$ incomplete moment about the origin of X with TG-QHR distribution is

$$\phi_r(t) = \int_0^t x^r f(x)dx$$

$$\phi_r(t) = \left[\left((1+\lambda)\theta W_{j,k,l,m} - 2\lambda\theta^2 W_{j,k+1,l,m}\right)\left(\frac{\alpha\Gamma(x; r+2l+3m+1)}{[\alpha(j+1)]^{(r+2l+3m+1)}} + \frac{\beta\Gamma(x; r+2l+3m+2)}{[\beta(j+1)]^{(r+2l+3m+2)}} + \frac{\gamma\Gamma(x; r+2l+3m+3)}{[\gamma(j+1)]^{(r+2l+3m+3)}}\right)\right]. \tag{23}$$

where Γ(x…) is the upper incomplete gamma function.

The mean deviation about mean is $MD_{\overline{X}} = E|X - \mu_1^1| = 2\mu_1^1 F(\mu_1^1) - 2\mu_1^1\phi_1$ and mean deviation about median is $MD_M = E|X - M| = 2MF(M) - 2M\phi_1$ where$\mu_1^{/} = E(X)\, and\, M = Q_{\frac{1}{2}}$. Bonferroni and Lorenz curves defined for a specified probability p by $\mathrm{B}(p) = \phi_1(q)/p\mu_1^1$ and $L(p) = \phi_1(q)/\mu_1^1$, where$q = Q_p$.

Residual life functions

The residual life, say $m_n(t)$, of X with TG-QHR distribution is

$$m_n(t) = E[(X - t)^n | X > t],$$

$$m_n(t) = \frac{1}{S(z)}\int_t^{\lambda} (x - t)^s f(x)dx,$$

$$m_n(t) = \frac{1}{(1 - F(t))}\sum_{r=0}^{n} \binom{n}{r}(-t)^{n-r}\phi_r(t),$$

$$m_n(t) = \frac{1}{(1-F(t))}\sum_{r=0}^{n}\binom{n}{r}(-t)^{n-r}\Bigg[\left((1+\lambda)\theta W_{j,k,l,m} - 2\lambda\theta^2 W_{j,k+1,l,m}\right) \times\left(\frac{\alpha\,\Gamma(x; r+2l+3m+1)}{[\alpha(j+1)]^{(r+2l+3m+1)}} + \frac{\beta\,\Gamma(x; r+2l+3m+2)}{[\beta(j+1)]^{(r+2l+3m+2)}} + \frac{\gamma\,\Gamma(x; r+2l+3m+3)}{[\gamma(j+1)]^{(r+2l+3m+3)}}\right)\Bigg].$$

The life expectancy or mean residual life function at a specified time t, say$m_1(t)$, quantifies the expected left over lifetime of an individual of age t and is given by.

$$m_1(t) = \frac{1}{(1-F(t))}\sum_{r=0}^{1}\binom{1}{r}(-t)^{1-r}\Bigg[\left((1+\lambda)\theta W_{j,k,l,m} - 2\lambda\theta^2 W_{j,k+1,l,m}\right) \times\left(\frac{\alpha\,\Gamma(x; r+2l+3m+1)}{[\alpha(j+1)]^{(r+2l+3m+1)}} + \frac{\beta\,\Gamma(x; r+2l+3m+2)}{[\beta(j+1)]^{(r+2l+3m+2)}} + \frac{\gamma\,\Gamma(x; r+2l+3m+3)}{[\gamma(j+1)]^{(r+2l+3m+3)}}\right)\Bigg].$$

The reverse residual life, say $M_n(t)$, of X with TG-QHR distribution is

$$M_n(t) = E[(t-X)^n / X \le t],$$

$$M_n(t) = \frac{1}{F(t)}\int_0^t (t-x)^n f(x)dx,$$

$$M_n(t) = \frac{1}{F(t)}\sum_{r=0}^{n}(-1)^r\binom{n}{r}t^{n-r}\phi_r(t),$$

$$M_n(t) = \frac{1}{F(t)}\sum_{r=0}^{n}(-1)^r\binom{n}{r}t^{n-r}\Bigg[\left((1+\lambda)\theta W_{j,k,l,m} - 2\lambda\theta^2 W_{j,k+1,l,m}\right) \times\left(\frac{\alpha\,\gamma(x; r+2l+3m+1)}{[\alpha(j+1)]^{(r+2l+3m+1)}} + \frac{\beta\,\gamma(x; r+2l+3m+2)}{[\beta(j+1)]^{(r+2l+3m+2)}} + \frac{\gamma\,\gamma(x; r+2l+3m+3)}{[\gamma(j+1)]^{(r+2l+3m+3)}}\right)\Bigg], \tag{24}$$

where $\gamma(x\ldots)$ is the lower incomplete gamma function.

The mean waiting time (MWT) or mean inactivity time signifies the waiting time passed since the failure of an item on condition that this failure had happened in the interval [0, t].The MWT of X, say $M_1(t)$, is defined by

$$M_1(t) = \frac{1}{F(t)}\sum_{r=0}^{1}(-1)^r\binom{1}{r}t^{1-r}\Bigg[\left((1+\lambda)\theta W_{j,k,l,m}-2\lambda\theta^2 W_{j,k+1,l,m}\right) \times\left(\frac{\alpha\gamma(x;r+2l+3m+1)}{[\alpha(j+1)]^{(r+2l+3m+1)}} + \frac{\beta\gamma(x;r+2l+3m+2)}{[\beta(j+1)]^{(r+2l+3m+2)}} + \gamma\frac{\gamma(x;r+2l+3m+3)}{[\gamma(j+1)]^{(r+2l+3m+3)}}\right)\Bigg]. \quad (25)$$

Characterizations

In order to develop a stochastic function in a certain problem, it is necessary to know whether the selected function fulfills the requirements of the specific underlying probability distribution. To this end, it is required to study characterizations of the specific probability distribution. Different characterization techniques have developed.

The TG-QHR distribution is characterized via (i) ratio of truncated Moments (ii) hazard function (iii) reverse hazard rate function and (iv) elasticity function.

Characterization of TG-QHR distribution via ratio of truncated moments

The TG-QHR distribution is characterized using Theorem 1 (Glänzel, 1986) on the basis of a simple relationship between two truncated moments of functions of X. Theorem 1 is given in Appendix A.

Proportion 5.1.1: Let $X:\Omega\rightarrow(0,\infty)$ be a continuous random variable. Let

$$h_1(x) = \theta^{-1}\left[1+(\theta-1)\left(1-e^{-Q(x|\alpha,\beta,\gamma)}\right)\right]^2\left[1+\lambda-\frac{2\lambda\theta\left(1-e^{-Q(x|\alpha,\beta,\gamma)}\right)}{(1+(\theta-1)(1-e^{-Q(x|\alpha,\beta,\gamma)}))}\right]^{-1}$$

and

$$h_2(x) = 2\theta^{-1}e^{-Q(x|\alpha,\beta,\gamma)}\left[1+(\theta-1)\left(1-e^{-Q(x|\alpha,\beta,\gamma)}\right)\right]^2\left[1+\lambda-\frac{2\lambda\theta\left(1-e^{-Q(x|\alpha,\beta,\gamma)}\right)}{(1+(\theta-1)(1-e^{-Q(x|\alpha,\beta,\gamma)}))}\right]^{-1}, x>0.$$

The pdf of X is (6) if and only if $p(x)$(in Theorem 1) has the form$p(x) = e^{Q(x|\alpha,\beta,\gamma)}$.

Proof. If X has pdf (6), then

$$(1-F(x))E(h_1(X)|X\geq x) = e^{-Q(x|\alpha,\beta,\gamma)}, x>0,$$

and

$$(1-F(x))E(h_2(X)|X\geq x) = e^{-2Q(x|\alpha,\beta,\gamma)}, x>0.$$

Conversely, if $p(x) = e^{Q(x|\alpha,\beta,\gamma)}$, then $p'(x) = A(x|\alpha,\beta,\gamma)\ e^{Q(x|\alpha,\beta,\gamma)}$ and the differential equation.

$s'(x) = \frac{p'(x)h_2(x)}{p(x)h_2(x)-h_1(x)} = 2A(x|\alpha,\beta,\gamma)$ has solution $s(x) = 2Q(x|\alpha,\beta,\gamma)$.

Therefore in the light of theorem 1, X has pdf (6).

Corollary 5.1.1: Let $X:\Omega \to (0,\infty)$be a continuous random variable and let

$$h_2(x) = 2\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)} \frac{\left[1+(\theta-1)\left(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\right)\right]^2}{\theta}\left[1+\lambda-\frac{2\lambda\theta\left(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\right)}{(1+(\theta-1)(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}))}\right]^{-1}, x>0.$$

The pdf of X is (6) if and only if there exist functions $p(x)$ *and* $h_1(x)$satisfy the equation

$$\frac{p'(x)}{p(x)h_2(x)-h_1(x)} = \theta\left(A(x|\alpha,\beta,\gamma)\mathrm{e}^{Q(x|\alpha,\beta,\gamma)}\right)\left[1+(\theta-1)\left(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\right)\right]^{-2} \left[1+\lambda-\frac{2\lambda\theta\left(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\right)}{(1+(\theta-1)(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}))}\right]. \tag{26}$$

Remark 5.1.1: The general solution of (26) is.

$$p(x) = \mathrm{e}^{2Q(x|\alpha,\beta,\gamma)}\int\left(-h_1(x)\frac{\theta\left(A(x|\alpha,\beta,\gamma)\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\right)}{[1+(\theta-1)(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)})]^{-2}} \left[1+\lambda-\frac{2\lambda\theta\left(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\right)}{(1+(\theta-1)(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}))}\right]\right)dx + D,$$

where D is a constant.

Characterization via hazard function

Here we characterize TG-QHR distribution via hazard function of X.

Definition 5.2.1: Let $X:\Omega \to (0,\infty)$ be a continuous random variable with cdf $F(x)$and pdf $f(x)$.The hazard function,$h_F(x)$, of a twice differentiable distribution function satisfies the differential equation

$$\frac{d}{dx}[\log f(x)] = \frac{h'_F(x)}{h_F(x)} - h_F(x).$$

Proposition 5.2.1 Let $X:\Omega \to (0,\infty)$ be continuous random variable. The pdf of X is (6), if and only if its hazard function, $h_F(x)$, satisfies the first order differential equation

$$h'(x)\mathrm{e}^{Q(x|\alpha,\beta,\gamma)} + A(x|\alpha,\beta,\gamma)\mathrm{e}^{Q(x|\alpha,\beta,\gamma)}h(x)$$
$$= \left[\frac{(\beta+2\gamma x)-(A(x|\alpha,\beta,\gamma))^2}{\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}} + \frac{\lambda(\beta+2\gamma x)\left(1-\lambda\left(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\right)\right)+\lambda^2(A(x|\alpha,\beta,\gamma))^2\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}}{\left(1-\lambda(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)})\right)^2}\right], x>0.$$

Proof: If the pdf of X is (6), then the above differential equation holds. Now, if the differential equation holds, then

$$\frac{d}{dx}\left\{h_F(x)\mathrm{e}^{Q(x|\alpha,\beta,\gamma)}\right\} = \frac{d}{dx}\left(A(x|\alpha,\beta,\gamma)\left[\frac{1}{\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}} + \frac{\lambda}{1-\lambda(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)})}\right]\right),$$

or

$$h(x) = A(x|\alpha,\beta,\gamma)\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\left[\frac{1}{\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}} + \frac{\lambda}{1-\lambda\left(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\right)}\right],$$

which is the hazard function of the TG-QHR distribution.

Characterization via reverse hazard function

Here we characterize TG-QHR distribution via reverse hazard function of X.

Definition 5.3.1: Let $X:\Omega\rightarrow(0,\infty)$ be a continuous random variable with cdf $F(x)$and pdf $f(x)$.The reverse hazard function,$r_F(x)$, of a twice differentiable distribution function satisfies the differential equation

$$\frac{d}{dx}[\log f(x)] = \frac{r'_F(x)}{r_F(x)} + r_F(x).$$

Proposition 5.3.1 Let $X:\Omega\rightarrow(0,\infty)$ be continuous random variable. The pdf of X is (6) if and only if its reverse hazard function,r_F, satisfies the first order differential equation.

$$r'_F(x)\mathrm{e}^{Q(x|\alpha,\beta,\gamma)} + A(x|\alpha,\beta,\gamma)\mathrm{e}^{Q(x|\alpha,\beta,\gamma)}r_F(x)$$

$$= \left[\frac{(\beta+2\gamma x)\left(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\right)-(A(x|\alpha,\beta,\gamma))^2\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}}{\left(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\right)^2} - \frac{\lambda(\beta+2\gamma x)\left(1+\lambda\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\right)+\lambda^2(A(x|\alpha,\beta,\gamma))^2\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}}{\left(1+\lambda\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\right)^2}\right].$$

Proof: If the pdf of X is (6), then the above differential equation holds. Now, if the differential equation holds, then

$$\frac{d}{dx}\left\{r_F(x)\mathrm{e}^{Q(x|\alpha,\beta,\gamma)}\right\} = \frac{d}{dx}\left[\frac{A(x|\alpha,\beta,\gamma)}{\left(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\right)} - \frac{\lambda A(x|\alpha,\beta,\gamma)}{1+\lambda\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}}\right],$$

or

$$r_F(x) = A(x|\alpha,\beta,\gamma)\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\left[\frac{1}{\left(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\right)} - \frac{\lambda}{1+\lambda\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}}\right], x>0$$

which is the reverse hazard function of the TG-QHR distribution.

Definition 5.4.1: Let $X:\Omega\rightarrow(0,\infty)$ be a continuous random variable with cdf $F(x)$ and pdf $f(x)$.The elasticity function, $e_F(x)$, of a twice differentiable distribution function satisfies the differential equation

$$\frac{d}{dx}[\ln f(x)] = \frac{e'(x)}{e(x)} + \frac{e(x)}{x} - \frac{1}{x}.$$

Proposition 5.4.1 Let $X:\Omega\rightarrow(0,\infty)$ be continuous random variable. The pdf of X is (6) if and only if its elasticity,$e_F(x)$, satisfies the first order differential equation.

$$e'_F(x)x\mathrm{e}^{Q(x|\alpha,\beta,\gamma)} + xA(x|\alpha,\beta,\gamma)\mathrm{e}^{Q(x|\alpha,\beta,\gamma)}e_F(x)$$

$$= \left[\frac{(\alpha x^{-1}+2\beta+3\gamma x)\left(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\right)+(A(x|\alpha,\beta,\gamma))^2\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}}{\left(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\right)^2} - \frac{\lambda(\alpha x^{-1}+2\beta+3\gamma x)\left(1+\lambda\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\right)+\lambda^2(A(x|\alpha,\beta,\gamma))^2\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}}{\left(1+\lambda\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\right)^2}\right].$$

Proof: If the pdf of X is (6), then the above differential equation holds. Now, if the differential equation holds, then

$$\frac{d}{dx}\left\{e_F(x)x\mathrm{e}^{Q(x|\alpha,\beta,\gamma)}\right\} = \frac{d}{dx}\left[A(x|\alpha,\beta,\gamma)\left[\frac{1}{\left(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\right)} - \frac{\lambda}{1+\lambda\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}}\right]\right],$$

or

$$e_F(x) = xA(x|\alpha,\beta,\gamma)\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\left[\frac{1}{\left(1-\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}\right)} - \frac{\lambda}{1+\lambda\mathrm{e}^{-Q(x|\alpha,\beta,\gamma)}}\right], x>0,$$

which is the elasticity function of the TG-QHR distribution.

Maximum likelihood estimation

In this section, parameters estimates are derived using maximum likelihood method. The log likelihood function for TG-QHR with the vector of parameters Φ = (α, β, θ, λ) is

$$L(x_i,\Phi) = n\ \ln\theta + \sum\ \ln A(x_i|\alpha,\beta,\gamma) - \sum Q(x_i|\alpha,\beta,\gamma) - 2 \sum\ \ln\left[1+(\theta-1)\left(1-e^{-Q(x_i|\alpha,\beta,\gamma)}\right)\right] + \sum\ \ln\left[1+\lambda-\frac{2\lambda\theta\left(1-e^{-Q(x_i|\alpha,\beta,\gamma)}\right)}{\left(1+(\theta-1)(1-e^{-Q(x_i\alpha,\beta,\gamma)})\right)}\right]. \tag{27}$$

In order to compute the estimates of parameters of TG-QHR distribution, the following nonlinear equations must be solved simultaneously:

$$\frac{\partial}{\partial\alpha}L(x_i,\Phi) = \sum(A(x_i|\alpha,\beta,\gamma))^{-1} - \sum x_i + 2\sum x_i\left[1+\frac{(\theta-1)^{-1}}{\left(1-e^{-Q(x_i|\alpha,\beta,\gamma)}\right)}\right]^{-1} + \frac{\lambda\theta}{2}\sum\left[\frac{x_i\left((\theta-1)+\left(1-e^{-Q(x_i|\alpha,\beta,\gamma)}\right)^{-1}\right)^{-2}\left(1-e^{-Q(x_i|\alpha,\beta,\gamma)}\right)^{-2}e^{-Q(x_i|\alpha,\beta,\gamma)}}{1+\lambda-2\lambda\theta\left((\theta-1)+\left(1-e^{-Q(x_i\alpha,\beta,\gamma)}\right)^{-1}\right)^{-1}}\right] = 0, \tag{28}$$

$$\frac{\partial}{\partial\beta}L(x_i,\Phi) = n\ \ln\theta + \sum\frac{x_i}{A(x_i|\alpha,\beta,\gamma)} - \frac{1}{2}\sum x_i^2 - (\theta-1)\sum\left[\frac{x^2e^{-Q(x_i|\alpha,\beta,\gamma)}}{1+(\theta-1)\left(1-e^{-Q(x_i|\alpha,\beta,\gamma)}\right)}\right] + \lambda\theta\sum\left[\frac{x_i^2\left((\theta-1)+\left(1-e^{-Q(x_i|\alpha,\beta,\gamma)}\right)^{-1}\right)^{-2}\left(1-e^{-Q(x_i|\alpha,\beta,\gamma)}\right)^{-2}e^{-Q(x_i\alpha,\beta,\gamma)}}{1+\lambda-2\lambda\theta\left((\theta-1)+\left(1-e^{-Q(x_i|\alpha,\beta,\gamma)}\right)^{-1}\right)^{-1}}\right] = 0, \tag{29}$$

$$\frac{\partial}{\partial\gamma}L(x_i,\Phi) = \sum\frac{x_i^2}{A(x_i|\alpha,\beta,\gamma)} - \frac{1}{3}\sum x_i^3 - \frac{2}{3}(\theta-1)\sum\left[\frac{x_i^3 e^{-Q(x_i\alpha,\beta,\gamma)}}{1+(\theta-1)(1-e^{-Q(x_i|\alpha,\beta,\gamma)})}\right]$$
$$+\frac{2\lambda\theta}{3}\sum\left[\frac{x_i^3\left((\theta-1)+\left(1-e^{-Q(x_i|\alpha,\beta,\gamma)}\right)^{-1}\right)^{-2}\left(1-e^{-Q(x_i|\alpha,\beta,\gamma)}\right)^{-2}e^{-Q(x_i|\alpha,\beta,\gamma)}}{1+\lambda-2\lambda\theta\left((\theta-1)+\left(1-e^{-Q(x_i|\alpha,\beta,\gamma)}\right)^{-1}\right)^{-1}}\right] = 0, \tag{30}$$

$$\frac{\delta}{\delta\lambda}L(x_i,\Phi) = \left[1-2\theta\left((\theta-1)+\left(1-e^{-Q(x_i|\alpha,\beta,\gamma)}\right)^{-1}\right)^{-1}\right]$$
$$\left[1+\lambda-2\lambda\theta\left((\theta-1)+\left(1-e^{-Q(x_i\alpha,\beta,\gamma)}\right)^{-1}\right)^{-1}\right]^{-1} = 0, \tag{31}$$

$$\frac{\delta}{\delta\theta}L(x_i,\Phi) = \frac{n}{\theta} - 2\sum\left[(\theta-1)+\left(1-e^{-Q(x_i\alpha,\beta,\gamma)}\right)^{-1}\right]^{-1}$$
$$-\sum\left[\frac{-2\lambda\left(\left(1-e^{-Q(x_i\alpha,\beta,\gamma)}\right)^{-1}+(\theta-1)\right)^{-1} - 2\lambda\theta\left(1-e^{-Q(x_i\alpha,\beta,\gamma)}\right)^{-1}\left(\left(1-e^{-Q(x_i|\alpha,\beta,\gamma)}\right)^{-1}+(\theta-1)\right)^{-2}}{1+\lambda-2\lambda\theta\left(\left(1-e^{-Q(x_i|\alpha,\beta,\gamma)}\right)^{-1}+(\theta-1)\right)^{-1}}\right] = 0. \tag{32}$$

Simulation studies

In this Section, we perform two simulation studies based on graphical results by using selected TG-QHR distributions. To see the performance of MLE's of these distributions, we generate $N = 1000$ samples of sizes $n = 20,30,\ldots,1000$ from TG-QHR distribution. For the first simulation study, we take true values of α, β, γ, θparameters as 5,3,2,0.5 respectively. For the second simulation study, we take true values of α, β, γ, θparameters as 2,2,1,0.2 respectively. Note that we assume λas known for both simulation studies. The random number generation is obtained with inverse of its cdf. The MLEs, say $(\hat{\alpha}_i, \hat{\beta}_i, \hat{\gamma}_i, \hat{\theta}_i)$for $i = 1,2,\ldots,N$, have been obtained by CG routine in R programme.

We also calculate the mean, standard deviations (sd), bias and mean square error (MSE) of the MLEs. The bias and MSE are calculated by (for $h = \alpha, \beta, \gamma, \theta$).

$$Bias_h = \frac{1}{1000}\sum_{i=1}^{1000}\left(\hat{h}_i - h\right) \text{ and } MSE_h = \frac{1}{1000}\sum_{i=1}^{1000}\left(\hat{h}_i - h\right)^2.$$

The results of the simulations are reported in Figs. 3 and 4. From these Figures, we can say that for four parameters the empirical means approach the true values when sample size increases whereas the bias, MSE and sd's decrease as the sample size increases, as expected.

Applications

The potentiality of TG-QHR distribution is demonstrated by its application to real data sets. The TG-QHR distribution is compared with G-QHR, T-QHR, QHR distributions. Goodness of fit of this distribution through different methods is studied. Different goodness fit measures such as "Cramer-von Mises (W), Anderson Darling (A), Kolmogorov-Smirnov (K-S) statistics with p-values and likelihood ratio

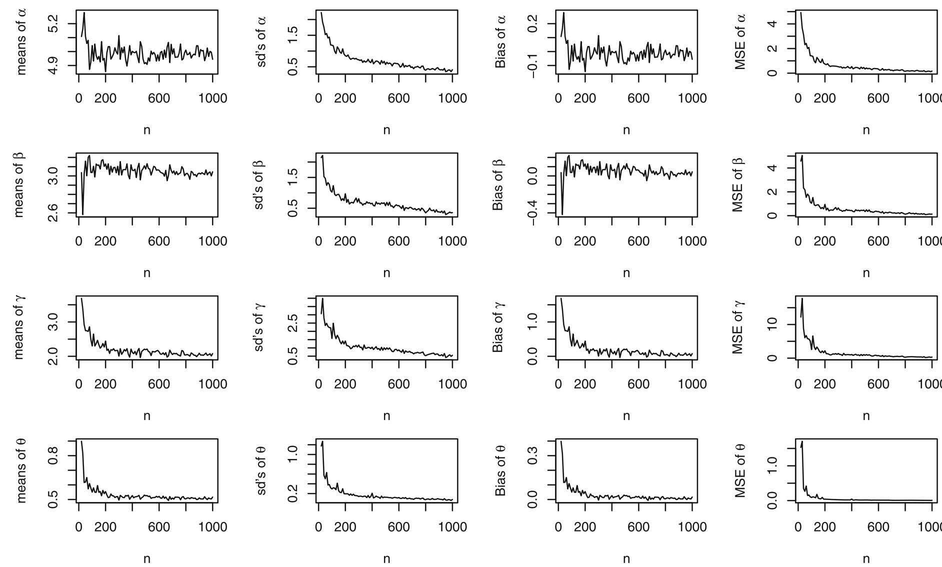

Fig. 3 First simulation results with $\lambda = -0.5$

statistics" are computed using R-package for strengths of 1.5 cm glass fibers and breaking stress of carbon fibers. The maximum likelihood estimates (MLEs) of unknown parameters and values of goodness of fit measures are computed for TG-QHR distribution and its sub-models. The better fit corresponds to smaller W, A, K-S and$-\ell$ value.

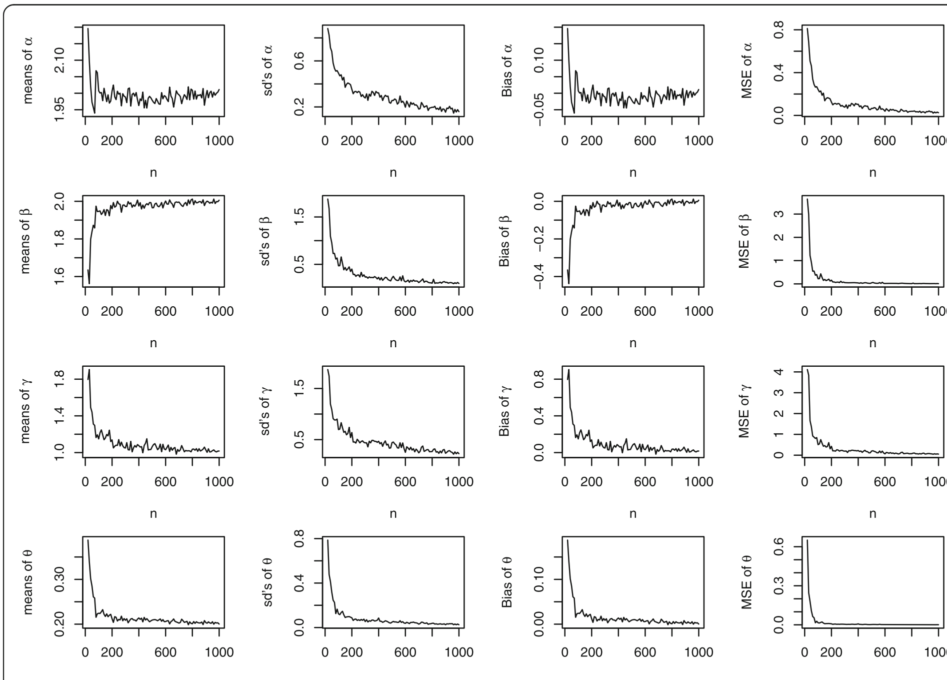

Fig. 4 Second simulation results with $\lambda = 0.5$

Table 2 MLEs (standard errors) and goodness-of-fit statistics for data set I

Model	α	β	γ	θ	λ
TG-QHR	0.0000000001	0.2887642399 (2.8360577)	2.0112261399 (1.6155598)	0.0286806160	0.7869934529 (0.2655901)
G-QHR	0.0000000001	0.0000000001	2.4729306644	0.0466476295	–
T-QHR	0.0000000001 (0.4973751)	0.0000000001 (1.3435556)	0.7774571279 (0.6531848)	–	0.0000000001
QHR	0.0000000001 (1.0627565)	0.0000000001 (2.1908955)	0.7774569066 (0.9845669)	–	–

Table 3 Goodness-of-fit statistics for data set I

Model	W	A	K-S (p-value)	−ℓ
TG-QHR	**0.08535245**	**0.4819065**	**0.0989 (0.5688)**	**11.54902**
G-QHR	0.1067002	0.5958679	0.1002 (0.5517)	12.05694
T-QHR	0.3824438	2.096342	0.2826 (8.512e-05)	30.84517
QHR	0.3824438	2.096342	0.2826 (8.512e-05)	30.84517

Boldface entries indicates that the proposed distribution is best fitted to data sets

Application I: Strengths of glass fibers

The values of data about strengths of 1.5 cm glass fibers (Smith and Naylor; 1987) are: 0.55, 0.74, 0.77, 0.81, 0.84, 0.93, 1.04, 1.11, 1.13, 1.24, 1.25, 1.27, 1.28, 1.29, 1.30, 1.36, 1.39, 1.42, 1.48, 1.48, 1.49, 1.49, 1.50, 1.50, 1.51, 1.52, 1.53, 1.54, 1.55, 1.55, 1.58, 1.59, 1.60, 1.61, 1.61, 1.61, 1.61, 1.62, 1.62, 1.63, 1.64, 1.66, 1.66, 1.66, 1.67, 1.68, 1.68, 1.69, 1.70, 1.70, 1.73, 1.76, 1.76, 1.77, 1.78, 1.81, 1.82, 1.84, 1.84, 1.89, 2.00, 2.01, 2.24.

The MLEs (standard errors) are given in Table 2. Table 3 displays goodness-of-fit statistics such as W, A, K-S (p-values) and −ℓ.

We can perceive that TG-QHR model is accurate fitted to data I because good of fit measures are smaller and graphical plots such as estimated pdf, cdf and pp. plots are closer to data set I (Fig. 5).

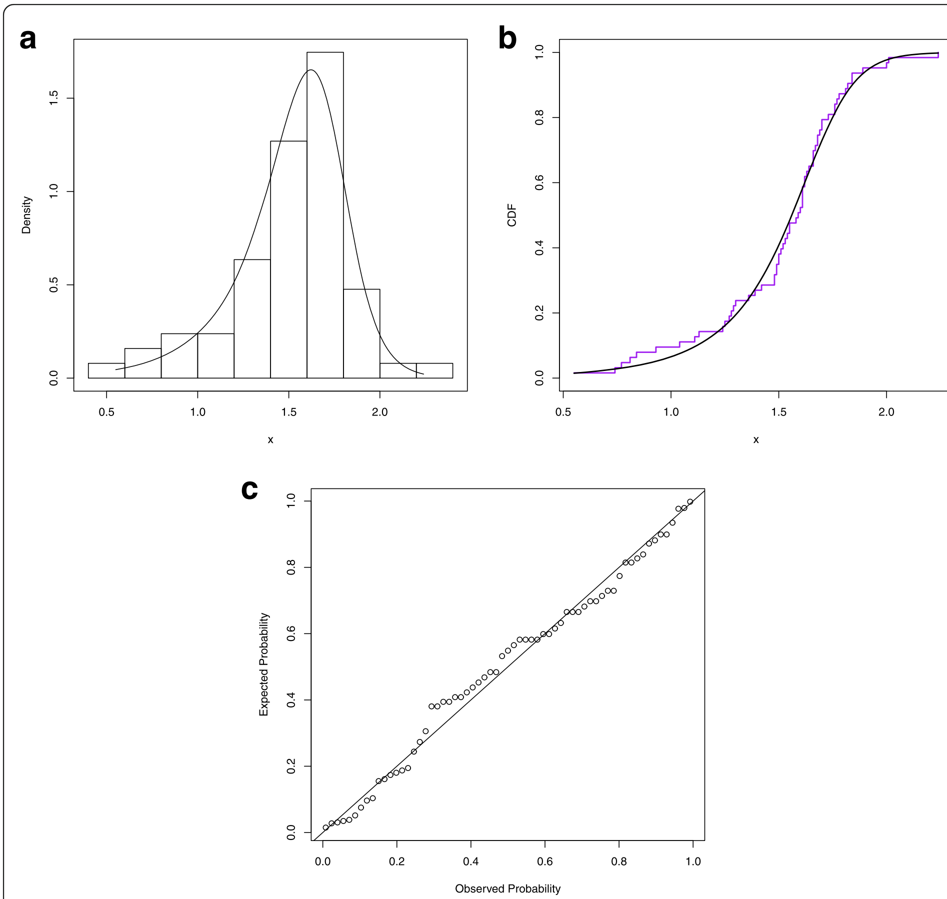

Fig. 5 Estimated pdf for TG-QHR and Estimated cdf for TG-QHR for data set I. PP plot for data Set I

Table 4 MLEs (standard errors) and goodness-of-fit statistics for data set II

Model	α	β	γ	θ	λ
TG-QHR	0.0000000001 (2.0716669)	0.5873823200 (1.5944635)	0.0000000001 (0.5562133)	0.0659761099 (0.5583977)	0.5373528912 (0.5614773)
G-QHR	0.0000000001 (1.1999485)	0.6277072331 (0.8641451)	0.0000000001 (0.3058022)	0.0927208734 (0.4125581)	
T-QHR	0.0000000001 (0.03431083)	0.0000000001 (0.06354651)	0.1094824490 (0.04205985)	–	0.0000000001
QHR	0.0000000001 (0.05259693)	0.0000000001 (0.13618412)	0.1094821345 (0.04713307)	–	–

Table 5 Goodness-of-fit statistics for data set II

Model	W	A	K-S	$-\ell$
TG-QHR	**0.03773332**	**0.2636783**	**0.0605 (0.9689)**	**84.68752**
G-QHR	0.04143493	0.2847517	0.0646 (0.9458)	84.8047
T-QHR	0.105313	0.5788886	0.1273 (0.2353)	87.01444
QHR	0.1053131	0.578889	0.1273 (0.2353)	87.01444

Boldface entries indicates that the proposed distribution is best fitted to data sets

Application II: Breaking stress of carbon fibers

The values of data about breaking stress of carbon fibers (Nichols and Padgett; 2006) are: 3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 3.56, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 1.57, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.89, 2.88, 2.82, 2.05, 3.65, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.35, 2.55, 2.59, 2.03, 1.61, 2.12, 3.15, 1.08, 2.56, 1.80, 2.53.

The MLEs (standard errors) are given in Table 4. Table 5 displays goodness-of-fit statistics such as W, A, K-S (p-values) and $-\ell$.

We can perceive that TG-QHR model is accurate fitted to data II because good of fit measures are smaller and graphical plots such as estimated pdf, cdf and pp. plots are closer to data set II (Fig. 6).

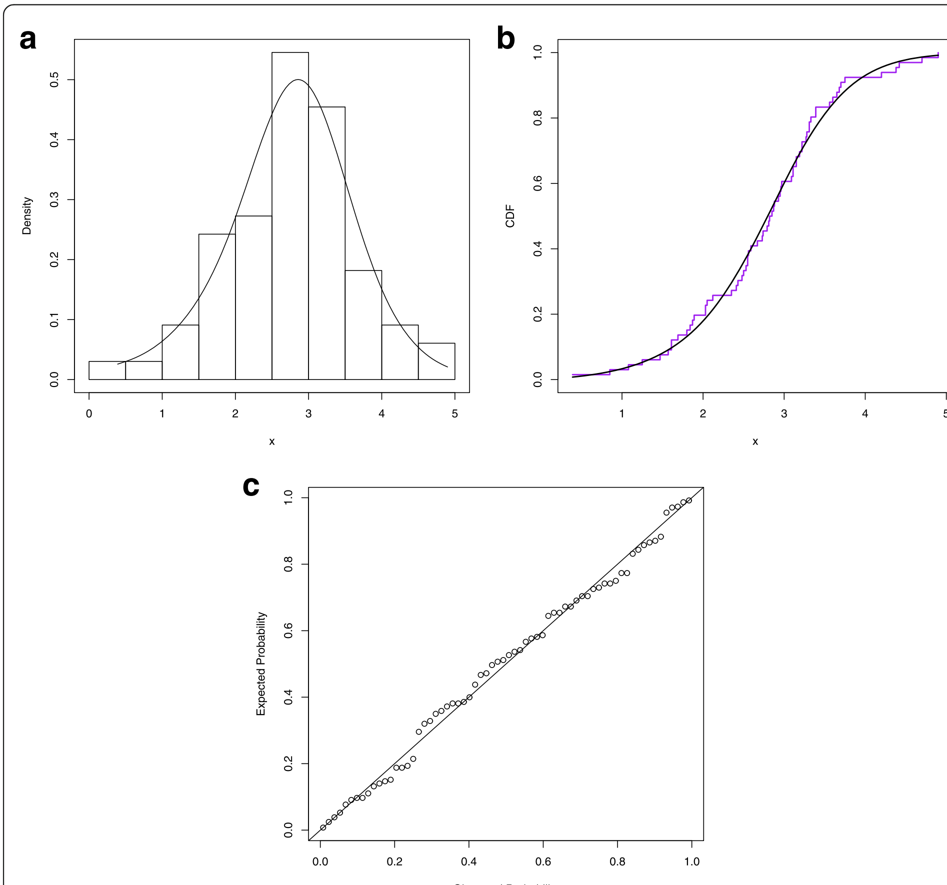

Fig. 6 Estimated pdf for TG-QHR and Estimated cdf for TG-QHR for data set II. PP Plots for data set II

Conclusions

We have developed a more flexible distribution which is suitable for applications in survival analysis, reliability and actuarial science. The important properties of the proposed TG-QHR distribution such as survival function, hazard function, reverse hazard function, cumulative hazard function, mills ratio, elasticity, quantile function, moments about the origin, incomplete moments and inequality measures are presented. The proposed distribution is characterized via different techniques. Maximum Likelihood estimates are computed. The simulation studies are performed on the basis of graphical results to illustrate the performance of maximum likelihood estimates of this distribution. Applications of our model to real data sets (strength of glass fibers and stress of carbon fibers) are given to show its significance and flexibility. Goodness of fit shows that our distribution is a better fit. We have demonstrated that the proposed distribution is empirically better for lifetime applications.

Appendix A

Theorem 1: *Let* $(\Omega, \mathrm{F}, \mathrm{P})$ *be a probability space and let* $[\mathrm{d}_1, \mathrm{d}_2]$ *be an interval with* $d_1 < d_2 d_1 = -\infty$, $d_2 = \infty$). *Also suppose that a continuous random variable* $X: \Omega \to [\mathrm{d}_1, \mathrm{d}_2]$ *has distribution function F. Let* $h_1(x)$ *and* $h_2(x)$ *be two real functions continuous on* $[\mathrm{d}_1, \mathrm{d}_2]$ *such that* $\frac{E[\mathrm{h}_1(X)|\mathrm{X} \geq \mathrm{x}]}{E[\mathrm{h}_2(X)|\mathrm{X} \geq \mathrm{x}]} = p(x)$ *where* $p(x)$ *is real function and should be in simple form. Assume that,* $h_1(x)$, $h_2\,(x) \varepsilon\, C([\mathrm{d}_1, \mathrm{d}_2]) p(x)\, \varepsilon\, C^2([\mathrm{d}_1, \mathrm{d}_2])$ *and F is strictly monotone function and twofold continuously differentiable on interval*$[\mathrm{d}_1, \mathrm{d}_2]$. *Assume that the equality* $h_2(x)p(x) = h_1(x)$ *has no real result inside of* $[\mathrm{d}_1, \mathrm{d}_2]$. *Then cdf* "F "*is obtained from* $h_1(x)$, $h_2(x)$ *and*$p(x)$*functions as* $F(x) = \int_0^x K\left|\frac{p'(t)}{p(t)h_2(t)-h_1(t)}\right| \exp(-s(t))dt$, *where* $s(t)$ *is obtained from equation* $s'(t) = \frac{p'(t)h_2(t)}{p(t)h_2(t)-h_1(t)}$ *and K is a constant, selected to make* $\int_{d_1}^{d_2} dF = 1$.

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Authors' contributions
FAB proposed the TG-QHR model and wrote the initial draft of the manuscript. MCK wrote graphs of skewness and kurtosis (Fig. 2), simulation Section "Applications" and graphs about applications (Figs. 5 and 6). Especially GGH did much work in order to improve the motivation, characterizations and revise the article as a whole. The authors, viz. FAB, GGH, MCK and MA with the consultation of each other finalized this work. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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