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Generalized logistic distribution and its regression model

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Abstract

A new generalized asymmetric logistic distribution is defined. In some cases, existing three parameter distributions provide poor fit to heavy tailed data sets. The proposed new distribution consists of only three parameters and is shown to fit a much wider range of heavy left and right tailed data when compared with various existing distributions. The new generalized distribution has logistic, maximum and minimum Gumbel distributions as sub-models. Some properties of the new distribution including mode, skewness, kurtosis, hazard function, and moments are studied. We propose the method of maximum likelihood to estimate the parameters and assess the finite sample size performance of the method. A generalized logistic regression model, based on the new distribution, is presented. Logistic-log-logistic regression, Weibull-extreme value regression and log-Fréchet regression are special cases of the generalized logistic regression model. The model is applied to fit failure time of a new insulation technique and the survival of a heart transplant study.

Keywords: Beta-family, Symmetric distribution, Hazard function, Moments, Censored data

2010 Mathematics subject classification: 62E15, 62F10, 62 J12, 62P10

Introduction

The use of logistic distribution in various disciplines can be found in (Johnson et al. 1995) and the references therein. The logistic distribution has the cumulative distribution function (CDF) defined as

$$F(x) = \left(1 + \exp\left(-\frac{x-\mu}{\sigma}\right)\right)^{-1}, \ -\infty < x, \mu < \infty, \sigma > 0. \tag{1}$$

Note that the logistic distribution is the limiting distribution of the average of largest and smallest values of random samples of size n from a symmetric distribution of exponential type (Gumbel 1958).

The CDF of the standard logistic distribution is $F(y) = (1 + e^{-y})^{-1}, \ -\infty < y < \infty$. The standard logistic density function with kurtosis 4.2 is symmetric about zero, and is more peaked and has heavier tails than the normal density function. These properties make logistic distribution a popular choice for fitting symmetric non-normal data.

The first type of extreme value distribution is commonly known as the Gumbel-type distribution due to Gumbel (1958), who made several significant contributions to the extreme value analysis and practical applications of extreme value statistics in distributions of human lifetimes, radioactive emissions, and flood analysis (see, e.g., Johnson et al. 1995). Gumbel used the distribution to model the maximum and minimum values of samples from various distributions. The CDFs of maximum and minimum Gumbel distributions are defined, respectively, as

$$F_{Gu\text{-max}}(x; \mu, \sigma) = \exp\left\{ -\exp\left(-\frac{x - \mu}{\sigma} \right) \right\}, \ -\infty < x < \infty, \ -\infty < \mu < \infty, \sigma > 0,$$
(2)

$$F_{Gu\text{-min}}(x; \mu, \sigma) = 1 - \exp\left\{ -\exp\left(\frac{x - \mu}{\sigma} \right) \right\}, \ -\infty < x < \infty, \ -\infty < \mu < \infty, \sigma > 0,$$
(3)

where μ and σ are location and scale parameters, respectively. Gumbel distribution is good to fit skewed data while logistic distribution is for symmetric data. It is interesting to note that there is a relation between these two distributions. If $X \sim \text{Gumbel}(\mu_X, \sigma)$ and $Y \sim \text{Gumbel}(\mu_Y, \sigma)$, then, $(X - Y) \sim \text{logistic}(\mu_X - \mu_Y, \sigma)$.

In order to improve the goodness of fit of the logistic and Gumbel distributions, many generalizations of these distributions have been studied in the literature. For example, Prentice (1976) proposed logistic type IV to model binomial regression data. Stukel (1988) proposed logistic regression model. Balakrishnan and Leung (1988) proposed three types of generalized logistic distribution. Johnson et al. (1995) summarized several generalizations of the logistic distribution. Wahed and Ali (2001) proposed the skew logistic distribution (SLD). An extension of SLD was presented and studied by Nadarajah (2009) by introducing a scale parameter. Gupta and Kundu (2010) defined two generalizations of logistic distribution, namely the skew logistic using the skew normal method proposed by Azzalini (1985) and defined the Type-II logistic distribution as a member of the proportional reversed hazard family with the baseline distribution as the logistic distribution. The *T-X* framework proposed by Alzaatreh et al. (2013), which was further expanded by Aljarrah et al. (2014) are two general methods that have been applied to derive various generalization of distributions, including logistic distribution. Recently, Ghosh and Alzaatreh (2018) defined the exponentiated-exponential logistic (EEL) distribution as a generalization of the logistic distribution and various properties were studied by the authors.

Similar to the logistic distribution, several generalizations of the Gumbel distribution have appeared in the literature. For a review of generalizations of the Gumbel extreme value distribution, one may refer to Pinheiro and Ferrari (2016).

There is already a long list of literatures for generalized logistic and Gumbel distributions. Why are we developing yet another family of generalized logistic distributions? As pointed out by Johnson et al. (1994, p. 15) "For most practical purposes, it is sufficient to use four parameters. There is no doubt that at least three parameters are needed; for some purposes this is enough." The main motivation is to develop highly flexible three-parameter distributions that can fit wide range of right and left skewed data. The method proposed here has several advantages that are not available among the existing generalizations:

(a) The method proposed is not to develop a single generalized logistic distribution, it can be applied to generate different families of generalized logistic distributions. A generalized normal distribution using similar technique was studied in Aljarrah et al. (2019), which was shown to be a much more flexible distribution than the skewed normal proposed by Azzalini (1985) and its generalizations.

(b) A member of the family of the generalized logistic distributions, the exponential-logistic {Generalized Weibull} distribution (E-L {GW}) is defined and studied in detail in this article. This distribution has three parameters: location, scale and a shape parameter. As shown in the article, the E-L {GW} distribution is a generalization of both logistic and Gumbel distributions.

(c) The E-L {GW} is shown to be more flexible than existing generalizations of logistic and Gumbel distributions in two ways: (i) It fits very well left and right skewed data. Existing generalized logistic or Gumbel distributions can fit heavy right-skewed data, but not able to fit heavy left-skewed data. (ii) It fits very well data with a wider range of skewness and kurtosis when compared with existing generalizations such as skew logistic (Gupta and Kundu 2010), beta-logistic distribution (Nassar and Elmasry 2012), generalized logistic distribution (Ghosh and Alzaatreh 2018), generalized Gumbel (Cooray 2010), as well as skew normal (Azzalini 1985) and its five-parameter generalized distribution (Choudhury and Abdul 2011).

(d) The generalized regression model derived by assuming the response follows E-L {GW} distribution is a very flexible model that takes logistic-log-logistic regression, Weibull-extreme value regression and log-Fréchet regression as special cases.

In Section 2, we define the E-L {GW} distribution. Some properties of the E-L {GW} distribution including the shapes of the probability density function (PDF) and hazard function, and quantile function are studied. An expression for the moment, properties of the hazard function, and the relationship between the mean, variance, skewness, kurtosis and the shape parameter are investigated in Section 3. In Section 4, the method of maximum likelihood is presented for estimating the parameters of the distribution, and a simulation study is performed to assess the small sample performance of the method. In Section 5, a generalized logistic regression model based on E-L {GW} distribution is developed. In Section 6, applications to several real data sets are given to demonstrate the flexibility and usefulness of the new distribution and its regression model. Summary and conclusions are given in Section 7.

The exponential-logistic {generalized Weibull} (E-L {GW}) distribution

Let the random variable R be a standard logistic distribution. Using a shape parameter $\xi > 0$, location parameter $-\infty < \mu < \infty$, scale reflection parameter $\sigma \neq 0$, and following the technique that Aljarrah et al. (2019) used to define the combined exponential-normal {GW} distribution, we define the combined E-R {GW} family as

$$F_X(x) = 0.5 + \text{sgn}(\sigma)\left(0.5 - \exp\left\{-\left(\left(\overline{F}_R\left(\frac{x-\mu}{\sigma}\right)\right)^{-\xi} - 1\right)/\xi\right\}\right), \qquad (4)$$

where sgn(σ) is the sign of the parameter σ. Note that the CDF defined in (4) reduces to $F_R\left(\frac{x-\mu}{|\sigma|}\right)$ distribution as $\xi \to 0$. The corresponding PDF to (4) is given by

$$f_X(x) = \frac{f_R\left(\frac{x-\mu}{\sigma}\right)}{|\sigma|\;\left(\overline{F}_R\left(\frac{x-\mu}{\sigma}\right)\right)^{\xi+1}}\;\exp\left(-\;\frac{\left(\overline{F}_R\left(\frac{x-\mu}{\sigma}\right)\right)^{-\xi}-1}{\xi}\right). \tag{5}$$

The E-L {GW} distribution can be defined from Eq. (4) by letting R be the logistic random variable as follows:

Definition (E-L {GW} distribution): The CDF and PDF of the E-L {GW} distribution are defined, respectively, as

$$F_X(x) = 0.5 + \;\text{sgn}(\sigma)\left(0.5 - \;\exp\left\{\left(1 - \left(1 + \;\exp\left(\frac{x-\mu}{\sigma}\right)\right)^{\xi}\right)\Big/\xi\right\}\right), \tag{6}$$

and

$$f_X(x) = \frac{1}{|\sigma|}\;\exp\left(\frac{x-\mu}{\sigma}\right)\left(1 + \;\exp\left(\frac{x-\mu}{\sigma}\right)\right)^{\xi-1}\exp\left\{\left(1 - \left(1 + \;\exp\left(\frac{x-\mu}{\sigma}\right)\right)^{\xi}\right)\Big/\xi\right\},$$
$$-\infty < x, \mu < \infty, \sigma \neq 0, \xi > 0. \tag{7}$$

Note the E-L {GW} is derived as a generalization of the symmetric logistic distribution for fitting highly skewed data. This provides a good comparison of performance when comparing with various existing three-parameter distributions. The following Corollary presents some special sub-models.

Corollary 1: The PDF of E-L{GW}(μ, σ, ξ) in (7) reduces to the following sub-models:

a) When $\xi \to 0$, the PDF in (7) reduces to a logistic distribution in (1).
b) When $\xi = 1$ and $\sigma < 0$, the PDF in (7) reduces to the PDF of maximum Gumbel distribution in (2) with location and scale parameters μ and $|\sigma|$, respectively.
c) When $\xi = 1$ and $\sigma > 0$, the PDF in (7) reduces to the PDF of minimum Gumbel distribution in (3) with location and scale parameters μ and σ, respectively.

Proof: a) $\lim_{\xi \to 0} f_X(x) = \frac{1}{|\sigma|}\;\exp\left(\frac{x-\mu}{|\sigma|}\right)\Big/\left(1 + \;\exp\left(\frac{x-\mu}{|\sigma|}\right)\right)^2$, that is $X \sim$ Logistic $(\mu, |\sigma|)$. The cases (b) and (c) are obtained directly by substituting $\xi = 1$ in (7). □

Quantile functions are useful for generating pseudo-random numbers from a probability distribution. Proposition 1 gives the quantile function for the E-L {GW} distribution.

Proposition 1: The quantile function for the E-L {GW} distribution is given by

$$Q_X(u) = \mu + \sigma\;\log\left(\left\{1 - \xi\;\log\left(\frac{1}{2} - \;\text{sgn}(\sigma)\left(u - \frac{1}{2}\right)\right)\right\}^{1/\xi} - 1\right), u \in (0, 1). \tag{8}$$

Proof: By setting $F_X(Q_X(u)) = u$ in Eq. (6) and solving for $Q_X(u)$ in terms of u, the quantile function in (8) is obtained. □

Proposition 2:

a) If T is a standard exponential random variable, then $X = \mu + \sigma \log((1 + \xi T)^{1/\xi} - 1)$
    follows the E-L {GW} (μ, σ, ξ) distribution in Eq. (6).
b) If $X \sim$ E-L {GW} (μ, σ, ξ), then $(2\mu - X) \sim$ E-L {GW} ($\mu, -\sigma, \xi$).

*Proof*: Using the CDF method, the results in (a) and (b) follow. □

The hazard rate function (HRF) of the E-L {GW} distribution is obtained after using the CDF in (6) and PDF in (7), and it is given by

$$
h(x) = \begin{cases} \dfrac{1}{\sigma}\exp\!\left(\dfrac{x-\mu}{\sigma}\right)\left\{\exp\!\left(\dfrac{x-\mu}{\sigma}\right)+1\right\}^{\xi-1}, & \sigma > 0, \\[4mm] \dfrac{\exp\!\left(\dfrac{x-\mu}{\sigma}\right)\left\{\exp\!\left(\dfrac{x-\mu}{\sigma}\right)+1\right\}^{\xi-1}}{\mid\sigma\mid\left\{\exp\!\left(\dfrac{1}{\xi}\left[\left\{\exp\!\left(\dfrac{x-\mu}{\sigma}\right)+1\right\}^{\xi}-1\right]\right)-1\right\}}, & \sigma < 0. \end{cases}
$$

$$(9)$$

Figures 1 and 2 show the plots of PDF and HRF for E-L {GW} distribution. The PDF can be positively or negatively skewed, while the HRF shows increasing with J shape, increasing with S shape, and increasing-decreasing shapes. The graphs in Fig. 1 indicate that the distribution tends to be symmetric as $\xi \to 0$, skewed to the left when $\sigma > 0$, and skewed to the right when $\sigma < 0$. When the sign of parameter σ is changed, the curve of the PDF is reflected about the line $x = 0$. Also as ξ increases, the mode decreases when $\sigma > 0$, and as ξ increases, the mode increases when $\sigma < 0$. The graphs in Fig. 2 show the hazard function in (9) is increasing when $\sigma > 0$. When $\sigma < 0$, the hazard function increases or first constant, increases and then decreases.

Properties of exponential-logistic {generalized Weibull} distribution

In this section, some properties of the E-L {GW} distribution are studied. These properties include, mode, shape property of the HRF, moments and moment generating function.
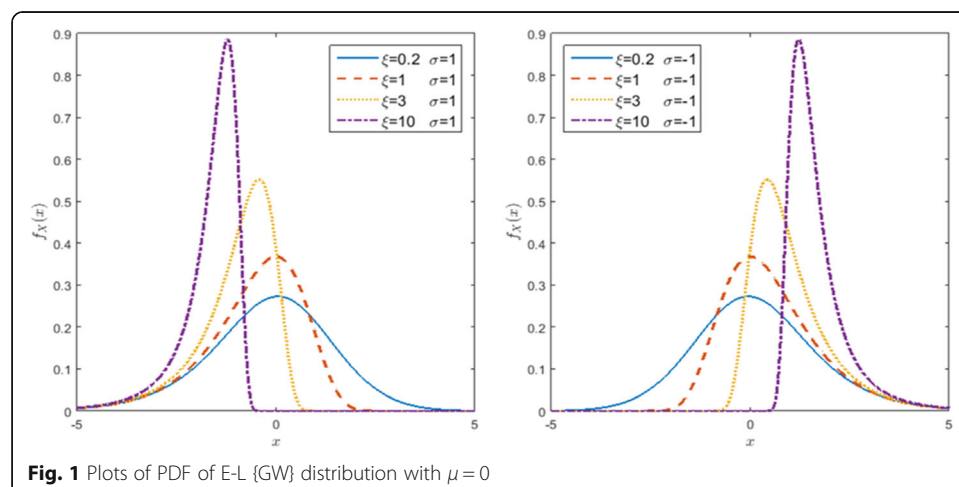
Fig. 1 Plots of PDF of E-L {GW} distribution with $\mu = 0$
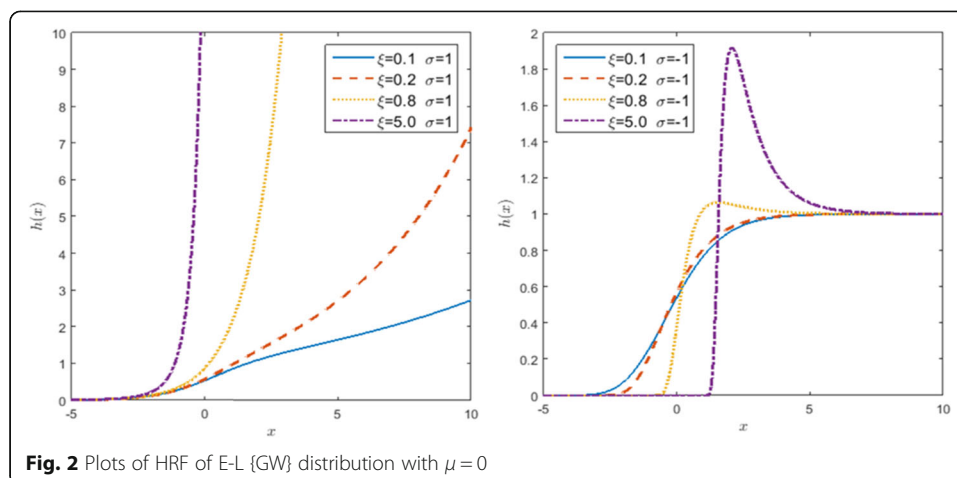
Fig. 2 Plots of HRF of E-L {GW} distribution with $\mu = 0$

Mode:

Theorem 1: *The E-L {GW} distribution is unimodal. The mode is at the point $x_* = \mu$ whenever $\xi = \{0, 1\}$. Otherwise the mode is at the point $x_* = \mu + \sigma \log(u_*)$, where u_* satisfies the equation*

$$\xi u + 1 = u(u + 1)^\xi, u > 0. \tag{10}$$

Proof: See Appendix.

Corollary 2: The HRF is increasing whenever $\sigma > 0$, and asymptotic to the line $y = 1/|\sigma|$ as $x \to \infty$ whenever $\sigma < 0$.

Proof: See Appendix.

It is noteworthy to mention that the graphs in Fig. 2 are consistent with the above results and the asymptotic feature of the curves in Corollary 2.

Moments: The moments are valuable for describing and identifying distribution properties such as the center, variance, skewness and kurtosis. In order to derive the moments of E-L {GW}, we first provide a series expansion of PDF of E-R {GW} in Eq. (5), by applying the exponential series, as follows.

$$f_X(x) = \frac{f_R\left(\frac{x-\mu}{\sigma}\right) \exp(1/\xi)}{|\sigma|} \sum_{i=0}^{\infty} \frac{(-1)^i \left(\overline{F}_R\left(\frac{x-\mu}{\sigma}\right)\right)^{-(\xi i + \xi + 1)}}{i! \xi^i}.$$

By applying negative binomial series expansion $(1-x)^{-r} = \sum_{j=0}^{\infty} \frac{\Gamma(r+j)}{\Gamma(j+1)\Gamma(r)} x^j$, $|x| < 1$ on $\left(\overline{F}_R((x-\mu)/\sigma)\right)^{-(\xi i + \xi + 1)}$, we get

$f_X(x) = \frac{f_R\left(\frac{x-\mu}{\sigma}\right)\exp(1/\xi)}{|\sigma|} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^i \Gamma(\xi i + \xi + j + 1)}{\xi^i i! \Gamma(j+1)\Gamma(\xi i + \xi + 1)} \left(F_R\left(\frac{x-\mu}{\sigma}\right)\right)^j$, which can be written as

$$f_X(x) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{\omega_{i,j}}{|\sigma|} k_{(j+1)}((x-\mu)/\sigma). \tag{11}$$

where

$$\omega_{i,j} = \frac{(-1)^i \exp(1/\xi)\Gamma(\xi i + \xi + j + 1)}{\xi^i i! \Gamma(j+2)\Gamma(\xi i + \xi + 1)}, \tag{12}$$

and $k_{(j+1)}(x) = (j+1)f_R(x)(F_R(x))^j$ denotes the PDF of exponentiated R random variable with power parameter $j + 1$.

Theorem 2: *The nth absolute moment of the E-L {GW} distribution exists for any μ, $\sigma \neq 0$, $\xi > 0$ and satisfies the inequality*

$$E(|X|^n) \leq e^{-1}(1+\xi)^{1+1/\xi} \sum_{i=0}^n \binom{n}{i} |\mu|^{n-i} |\sigma|^i E(|L|^i), \tag{13}$$

where L is a standard logistic random variable.

Proof: See Appendix.

Moments of E-L {GW} as a series expression is given in the following theorem.

Theorem 3: *The rth moment, $E(X^r)$, of the E-L {GW} distribution is given by*

$$E(X^r) = \sum_{n=0}^r \sum_{i=0}^\infty \sum_{j=0}^\infty \binom{r}{n} \mu^{r-n} \sigma^n \omega_{i,j} E\left(L_{j+1}^n\right), \tag{14}$$

where $\omega_{i,j}$ is defined in (12) and $E(L_{i+1}^n)$ is the nth moment of the exponentiated logistic distribution with power parameter $j + 1$ and given by Ali et al. (2007) as

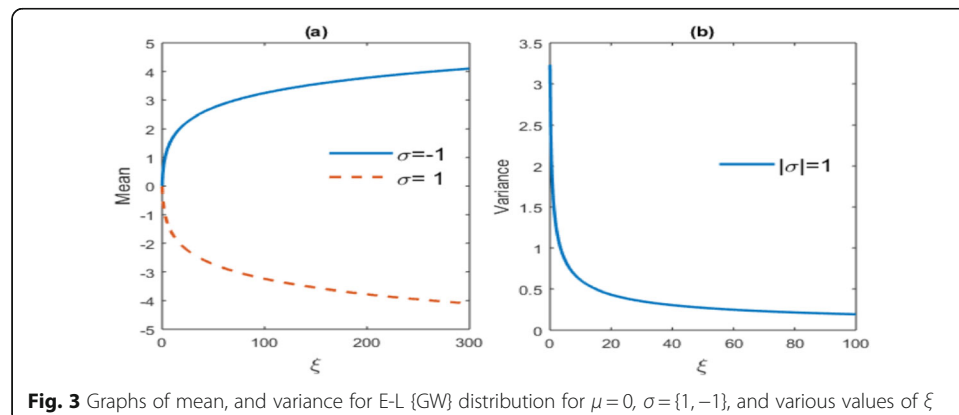$$E\left(L_{j+1}^n\right) = (j+1)n! \left(\sum_{k=0}^\infty \frac{(-j-2)P_k}{k!(k+1)^{n+1}} + (-1)^n \sum_{k=0}^\infty \frac{(-j-2)P_k}{k!(k+j+1)^{n+1}} \right).$$
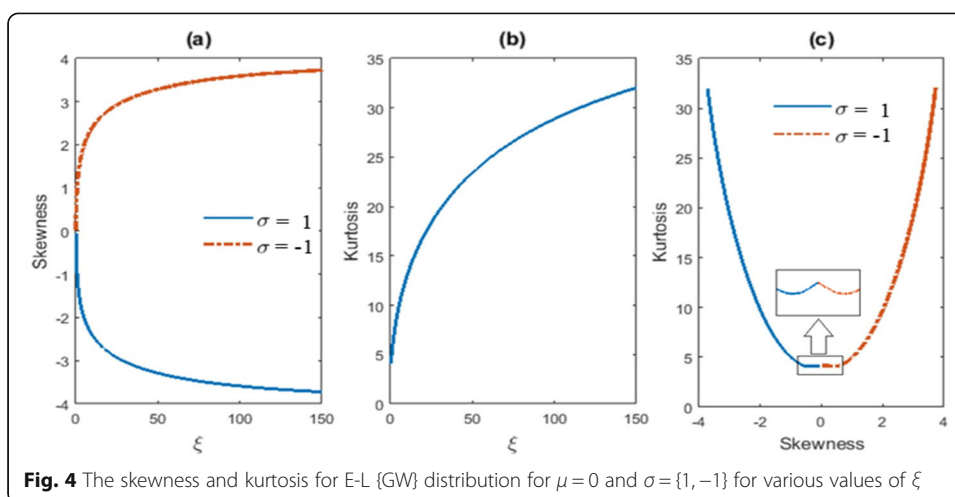
Proof: See Appendix.

Proposition 3: Suppose X has the PDF in (6), then the moment generating function (MGF) of X is given by

$$M_X(t) = e^{\mu t + 1/\xi} \sum_{i=0}^\infty \frac{\Gamma(\sigma t + 1)(-1)^i}{\Gamma(\sigma t - i + 1)\Gamma(i+1)} \xi^{(\sigma t - i)/\xi} \Gamma((\sigma t - i)/\xi + 1, 1/\xi), \tag{15}$$

where

Fig. 3 Graphs of mean, and variance for E-L {GW} distribution for $\mu = 0$, $\sigma = \{1, -1\}$, and various values of ξ

Fig. 4 The skewness and kurtosis for E-L {GW} distribution for $\mu = 0$ and $\sigma = \{1, -1\}$ for various values of ξ

$$\begin{cases} t \in (-\infty, 1/|\sigma|), & \xi \geq 1, \sigma < 0 \\ t \in (-1/\sigma, \infty), & \xi \geq 1, \sigma > 0 \\ t \in (-1/|\sigma|, 1/|\sigma|), & \xi < 1, \sigma \neq 0. \end{cases}$$

Proof: See Appendix.

In Fig. 3, the mean and variance of E-L {GW} distribution are plotted in terms of the parameter ξ for $\mu = 0$ and $\sigma = \{1, -1\}$. Figure 3(a) shows that when $\sigma > 0$, the mean decreases as ξ increases, When $\sigma < 0$, the mean increases as ξ increases. Also, Fig. 3(b) shows that the variance decreases as ξ increases.

In Fig. 4, we plot the skewness and kurtosis of E-L {GW} distribution in terms of the parameters ξ when $\mu = 0$ and $\sigma = \{1, -1\}$. Figure 4(a) shows that when $\sigma > 0$, the skewness decreases as ξ increases and the E-L {GW} distribution is left skewed, and when $\sigma < 0$, the skewness increases as ξ increases and the E-L {GW} distribution is right skewed. The distribution is symmetric as $\xi \rightarrow 0$. We note that the degree of skewness of the E-L {GW} distribution is measured by ξ, and the parameter σ plays two roles: characterizing the scale property and determining left skewed ($\sigma > 0$) or right skewed ($\sigma < 0$). Figure 4(b) shows the kurtosis increases as ξ increases, and it is not affected by σ.

Table 1 A comparison of skewness and kurtosis of some generalized logistic and Gumbel distributions

Distribution	Skewness	Kurtosis
SN	−0.995 to 0.995	3.000 to 3.869
ESGN	−0.995 to 0.995	3.000 to 7.238
GN	−3.000 to 3.000	3.000 to 20.000
BGL	−2.000 to 2.000	3.000 to 6.000
PRHL	−2.000 to 1.140	1.138 to 6.000
GG	−1.500 to 1.500	4.245 to 6.200
EEL	−1.080 to 1.889	2.393 to 8.553
E-L {GW}	−4.000 to 4.000	3.956 to 35.000

The flexibility of the E-L {GW} is compared with skew normal (SN) (Azzalini 1985), extended skew generalized normal (ESGN) (Choudhury and Abdul 2011), generalized normal (GN) (Aljarrah et al. 2019), beta-generalized logistic (BGL) (Nassar and Elmasry 2012), proportional reversed hazard logistic (PRHL) (Gupta and Kundu 2010), generalized Gumbel (GG) (Cooray 2010) and EEL (Ghosh and Alzaatreh 2018). Table 1 summarizes the ranges of skewness and kurtosis of these distributions. It is shown that the E-L {GW} fits the widest range of skewness and kurtosis with the exception that the PRHL can fit platykurtic distributions.

Estimation and simulation

Estimation

In this subsection, we discuss the maximum likelihood estimation method for the parameters of E-L {GW} distribution. Let $x_1, x_2, ..., x_n$ be a random sample from E-L {GW} distribution with parameters $\boldsymbol{\theta} = (\xi, \mu, \sigma)^t$, the log-likelihood function is given by

$$\ell(\boldsymbol{\theta}, \boldsymbol{x}) = -n \log |\sigma| + \sum_{i=1}^{n} \left(\frac{x_i - \mu}{\sigma}\right) + (\xi - 1) \sum_{i=1}^{n} \log \left[1 + \exp\left(\frac{x_i - \mu}{\sigma}\right)\right]$$
$$+ \frac{n}{\xi} - \frac{1}{\xi} \sum_{i=1}^{n} \left[1 + \exp\left(\frac{x_i - \mu}{\sigma}\right)\right]^{\xi}.$$

Letting $z_i = \exp((x_i - \mu)/\sigma)$, the score function of the distribution parameters is given by $U_n(\boldsymbol{\theta}) = (\partial\ell/\partial\xi, \partial\ell/\partial\mu, \partial\ell/\partial\sigma)$, where

$$\frac{\partial \ell}{\partial \xi} = -\frac{1}{\xi} \sum_{i=1}^{n} \left[(z_i + 1)^{\xi} \log(z_i + 1)\right] + \frac{1}{\xi^2} \sum_{i=1}^{n} [z_i + 1]^{\xi} - \frac{n}{\xi^2} + \sum_{i=1}^{n} \log(z_i + 1), \quad (16)$$

$$\frac{\partial \ell}{\partial \mu} = \frac{1}{\sigma} \sum_{i=1}^{n} \left[z_i(z_i + 1)^{\xi - 1}\right] - \frac{1}{\sigma} \sum_{i=1}^{n} \frac{\xi z_i + 1}{z_i + 1}, \quad (17)$$

$$\frac{\partial \ell}{\partial \sigma} = \frac{1}{\sigma} \sum_{i=1}^{n} \left[z_i \log z_i (z_i + 1)^{\xi - 1}\right] - \sum_{i=1}^{n} \frac{(1 + \xi \log z_i) z_i + 1 + \log z_i}{\sigma(z_i + 1)}. \quad (18)$$

The maximum likelihood estimates (MLEs) of the parameters can be obtained by solving the nonlinear Eqs. (16), (17) and (18). The initial values of μ and σ are taken to be the mean and ± standard deviation of the data respectively. The initial value of σ is taken as s (or -s) if the data is skewed left (or right). The initial value of ξ is taken to be 1.

Simulation

A simulation study is conducted to explore the performance of the MLE for the parameters of the E-L {GW} distribution. Many combinations of the parameters of the E-L {GW} model, namely, highly, moderately, and weakly left (or right) skewed, are considered and represent all possible shapes of the model. Different sample sizes $n = \{50, 100, 200, 500, 1000\}$ are also considered. The MLE of the parameters ξ, μ and σ are computed for 200 repetitions in order to calculate the bias and the standard deviation (SD) for each set of parameter combinations and sample size. Table 2 shows the results of the simulation, and Figs. 5 and 6 present the illustrations. The results show that the bias and SD decrease as the sample size increases. The estimated PDF curve also moves
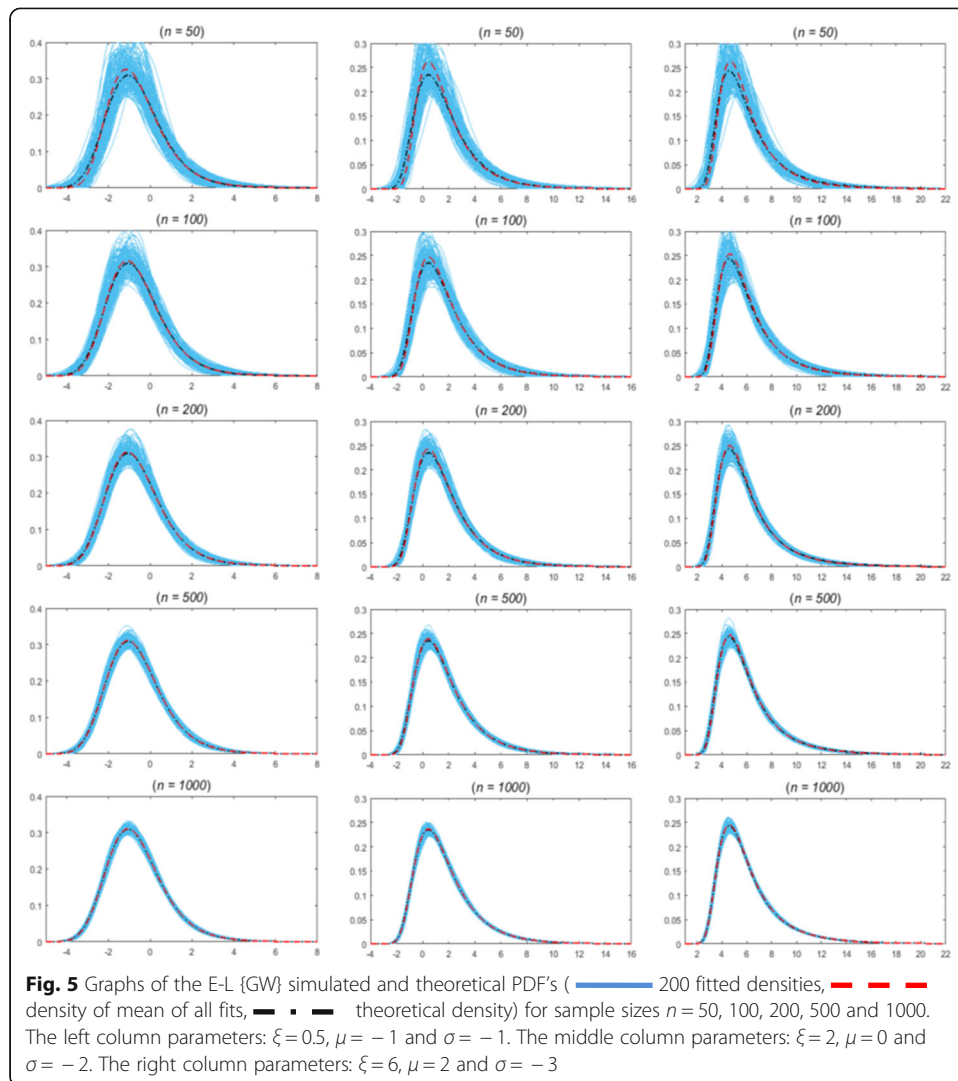
Table 2 Bias and SD of the parameter estimates using MLE method

Parameters			Sample size	Bias			SD		
ξ	μ	σ		$\hat{\xi} - \xi$	$\hat{\mu} - \mu$	$\hat{\sigma} - \sigma$	$SD(\hat{\xi})$	$SD(\hat{\mu})$	$SD(\hat{\sigma})$
0.5	−1	−1	50	0.2585	−0.1125	−0.0425	0.5947	0.3552	0.2162
			100	0.0979	−0.0386	−0.0160	0.3570	0.2413	0.1550
			200	0.0654	−0.0278	−0.0142	0.2278	0.1587	0.1187
			500	0.0265	−0.0070	−0.0050	0.1357	0.0999	0.0687
			1000	0.0102	0.0003	−0.0010	0.0928	0.0662	0.0474
2	0	−2	50	1.0074	−0.4525	−0.1085	2.5277	1.3595	0.6366
			100	0.3699	−0.1655	−0.0345	1.4639	0.8721	0.4376
			200	0.3072	−0.1583	−0.0494	1.1224	0.6916	0.3598
			500	0.1615	−0.0767	−0.0279	0.8690	0.5048	0.2375
			1000	0.0439	−0.0156	−0.0061	0.4078	0.2709	0.1442
6	2	−3	50	−1.3031	0.9211	0.4806	2.9698	1.5575	0.7229
			100	−0.8896	0.6087	0.2896	2.6342	1.3797	0.6079
			200	−0.4115	0.3154	0.1566	2.2743	1.1780	0.5040
			500	−0.4251	0.2871	0.1219	1.8211	0.9216	0.3595
			1000	−0.4952	0.2921	0.1137	1.3108	0.6636	0.2506
0.5	−1	1	50	0.2017	0.0813	0.0291	0.5106	0.3185	0.2133
			100	0.0981	0.0387	0.0161	0.3569	0.2412	0.1550
			200	0.0659	0.0283	0.0141	0.2289	0.1591	0.1189
			500	0.0271	0.0074	0.0054	0.1364	0.1004	0.0688
			1000	0.0116	0.0004	0.0018	0.0920	0.0662	0.0475
2	0	2	50	0.9262	0.4091	0.0931	2.3871	1.2927	0.6084
			100	0.3652	0.1627	0.0334	1.4761	0.8769	0.4393
			200	0.3072	0.1585	0.0513	1.1224	0.6939	0.3609
			500	0.1562	0.0713	0.0251	0.9162	0.5125	0.2358
			1000	0.0365	0.0106	0.0034	0.4089	0.2720	0.1444
6	4	3	50	1.0664	0.0219	−0.1725	6.0035	2.5586	0.9517
			100	0.8875	0.1094	−0.0699	5.0824	2.2413	0.8278
			200	0.6818	0.1183	−0.0310	3.9748	1.7626	0.6446
			500	0.4806	0.0767	−0.0105	3.7760	1.6030	0.5314
			1000	0.1190	−0.0384	−0.0374	2.7782	1.2040	0.3957

closer to the actual curve with the increase in the sample size. These results indicate that the MLE method can be used to estimate the parameters of the E-L {GW} distribution.

Generalized logistic regression model based on E-L {GW}

In this section, we propose a generalized logistic regression model by assuming the response Y follows E-L {GW} distribution. If the variable of interest is non-negative such as survival time, T, then the response Y is defined as $\log(T)$. In the following, we derive a generalized logistic regression model for modeling life-time data. Univariate survival functions and censored data regression problems can be estimated using parametric models for covariate effects. Parametric models produce precise estimates of the quantities of interest when they provide a good fit to the lifetime data set. The reason is that
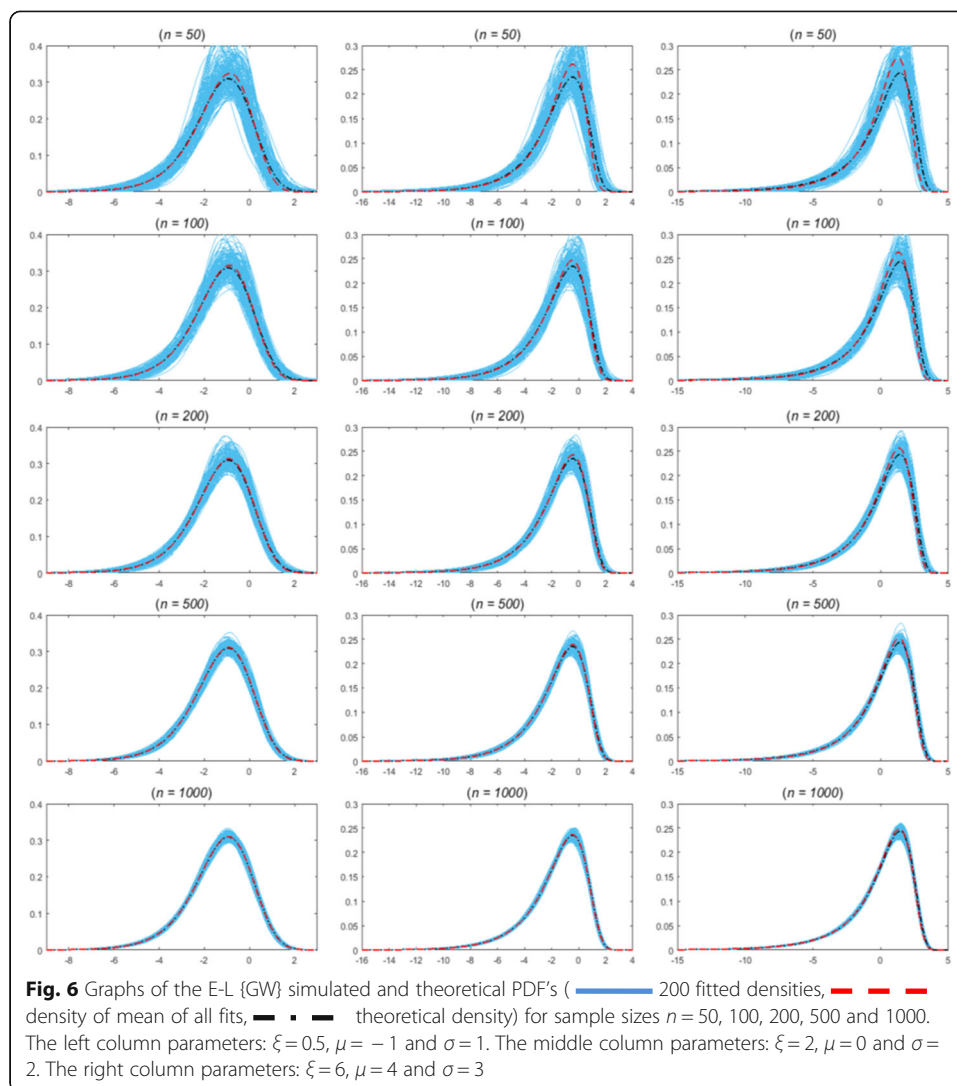
Fig. 5 Graphs of the E-L {GW} simulated and theoretical PDF's (▬▬▬▬ 200 fitted densities, ▬ ▬ ▬ density of mean of all fits, ▬ · ▬ theoretical density) for sample sizes $n = 50, 100, 200, 500$ and 1000. The left column parameters: $\xi = 0.5, \mu = -1$ and $\sigma = -1$. The middle column parameters: $\xi = 2, \mu = 0$ and $\sigma = -2$. The right column parameters: $\xi = 6, \mu = 2$ and $\sigma = -3$

these estimates are based on few parameters in this way. On the basis of the E-L {GW} distribution, the following regression model is considered:

$$y_i = \mathbf{v}_i^T \boldsymbol{\beta} + \sigma z_i, i = 1, ..., n, \tag{19}$$

where the response variable $y_i = \log(t_i)$ is the logarithm of the survival time t_i, $\boldsymbol{\beta} = (\beta_0, \beta_1, ..., \beta_p)^T$, and $\sigma \neq 0$ are unknown parameters. Each y_i has a covariate vector $\mathbf{v}_i^T = (1, v_{i1}, ..., v_{ip})$ that models the linear predictor $\mu_i = \mathbf{v}_i^T \boldsymbol{\beta}$. The random error z_i has the E-L {GW} density (7). The shape parameter ξ can be treated as a nuisance parameter, which may be tested against special cases of the E-L {GW} distribution. It can also be modeled with a vector of covariates $\xi_i = \exp(\mathbf{v}_i^T \boldsymbol{\gamma})$ that depends on the covariate vector $\mathbf{v}_i^T$ and parameter vector $\boldsymbol{\gamma} = (\gamma_0, \gamma_1, ..., \gamma_p)^T$. The corresponding survival function is

$$S(y_i | \mu(\mathbf{v}), \sigma, \xi(\mathbf{v})) = 0.5 - \operatorname{sgn}(\sigma) \left(0.5 - \exp\left\{ \left(1 - \left(1 + \exp\left(\frac{y_i - \mathbf{v}_i^T \boldsymbol{\beta}}{\sigma} \right) \right)^{\exp(\mathbf{v}_i^T \boldsymbol{\gamma})} \right) / \exp(\mathbf{v}_i^T \boldsymbol{\gamma}) \right\} \right). \tag{20}$$

The corresponding PDF to the survival function in (20) is given by

Fig. 6 Graphs of the E-L {GW} simulated and theoretical PDF's (══════ 200 fitted densities, ▬ ▬ ▬ density of mean of all fits, ▬ • ▬ theoretical density) for sample sizes $n = 50, 100, 200, 500$ and 1000. The left column parameters: $\xi = 0.5$, $\mu = -1$ and $\sigma = 1$. The middle column parameters: $\xi = 2$, $\mu = 0$ and $\sigma = 2$. The right column parameters: $\xi = 6$, $\mu = 4$ and $\sigma = 3$

$$
f(y_i) = \frac{1}{|\sigma|} \exp\left(\frac{y_i - v_i^T \boldsymbol{\beta}}{\sigma}\right) \left(1 + \exp\left(\frac{y_i - v_i^T \boldsymbol{\beta}}{\sigma}\right)\right)^{\exp(v_i^T \boldsymbol{\gamma}) - 1} \exp\left\{\left(1 - \left(1 + \exp\left(\frac{y_i - v_i^T \boldsymbol{\beta}}{\sigma}\right)\right)^{\exp(v_i^T \boldsymbol{\gamma})}\right) \middle/ \exp(v_i^T \boldsymbol{\gamma})\right\}.
$$

$$(21)$$

The generalized logistic regression model consists of many popular regression models as nested models. Some special regression models are as follows:

1. Logistic-log-logistic regression model: this model is obtained as a special case from (20) when $\gamma_1 = \gamma_1 = \ldots = \gamma_p = 0$ and $\gamma_0 \to -\infty$ (or $\xi \to 0$). The survival function is

$$
S(y) = \left\{1 + \exp\left(\frac{y - v^T \boldsymbol{\beta}}{|\sigma|}\right)\right\}^{-1},
$$

which is the logistic-log-logistic regression model, Lawless (2003, p. 303).

2. Weibull-extreme value regression model: this model is obtained as a special case from (20) when $\gamma_0 = \gamma_1 = \ldots = \gamma_p = 0$ (or $\xi = 1$), and $\sigma > 0$. The survival function is

$$S(y) = \exp\left\{ -\exp\left(\frac{y - \boldsymbol{v}^T\boldsymbol{\beta}}{\sigma} \right) \right\},$$

which is the classical Weibull regression model, Lawless (2003, p. 296).

3. Log-Fréchet regression model: this model is obtained as a special case from (20) when $\gamma_0 = \gamma_1 = \ldots = \gamma_p = 0$ (or $\xi = 1$), and $\sigma < 0$. The survival function is

$$S(y) = 1 - \exp\left\{ \exp\left(-\frac{y - \boldsymbol{v}^T\boldsymbol{\beta}}{|\sigma|} \right) \right\},$$

which is the log-Fréchet regression model (Alamoudi et al. 2017).

A sample $(y_1, \boldsymbol{v}_1), \ldots, (y_n, \boldsymbol{v}_n)$ of n independent observations is considered, where each random response is defined by $y_i = \min\{\log(t_i), \log(c_i)\}$, where c_i is the censoring time. We assume non-informative censoring and independent observed lifetimes and censoring times. Let Ω and C denote the sets of individuals for which y_i is the log-lifetime and log-censoring respectively. The total log-likelihood function for the model parameters $\boldsymbol{\theta} = (\sigma, \boldsymbol{\beta}^T, \boldsymbol{\gamma}^T)^T$ is given as

$$\ell(\boldsymbol{\theta}) = \sum_{i \in \Omega} \log[f(y_i)] + \sum_{i \in C} \log[S(y_i)], \tag{22}$$

where $S(y_i)$ is the survival function in (20) and $f(y_i)$ is the PDF of $S(y_i)$ in (21). The MLE $\hat{\boldsymbol{\theta}}$ of the parameter vector $\boldsymbol{\theta} = (\sigma, \boldsymbol{\beta}^T, \boldsymbol{\gamma}^T)^T$ of the E-L {GW} regression model can be obtained by maximizing the log-likelihood function in (22).

Applications

In this section, we apply the E-L {GW} distribution to fit two skewed data and apply the generalized logistic regression to model two censored lifetime data. For the first two data sets, the fits of the E-L {GW} distribution are compared with those of other recent generalizations of logistic and Gumbel distributions, namely, the EEL distribution by Ghosh and Alzaatreh (2018), PRHL distribution by Gupta and Kundu (2010), GG by Cooray (2010), and transmuted extreme value (TEV) by Aryal and Tsokos (2009). Maximum likelihood method is used to estimate the model parameters in these applications.

The fitted distributions are compared by using the Akaike information criterion (AIC) and Kolmogorov-Smirnov (KS) statistic and its *p*-value. Data have a good fit when the values of AIC and KS are small, and the *p*-value of KS is large. The plots of the fitted PDFs of some models are demonstrated for visual comparison. Table 3 gives the descriptive statistics of the two data sets. For the third and fourth applications, the generalized logistic regression models are compared with some nested sub-models. The goodness of fits are compared using AIC, the corrected AIC (AICC), and Bayesian

Table 3 The summary statistics of the data sets

Data set	No. of Obs.	mean	Std. Dev.	Skewness	Kurtosis
Adiponectin	116	10.1809	6.8433	1.7942	6.7108
Turbocharger	40	6.2525	1.9555	−0.6626	2.6410

information criterion (BIC) statistics. The estimation process is straightforward, and the R programming language is used for the first two data sets, while SAS programming language is used for the third and fourth data sets.

Adiponectin data

The data consist of 116 measurements of Adiponectin from Patrício et al. (2018). The data set is fitted to the E-L {GW} model presented in Section 2 and EEL, PRHL, GG, and TEV distributions. Table 4 indicates that the p-values of KS statistics of the distributions provide adequate fit to the data. While the five distributions all have three parameters, E-L {GW} provides the best fit to the data set. Therefore, the E-L {GW} distribution is a better alternate distribution to EEL, PRHL, GG, and TEV distributions. The large skewness and kurtosis of the sample data in Table 3 and the wide range of theoretical skewness and kurtosis in Table 1 suggest that E-L {GW} should fit better than other comparable distributions. Figure 7 shows the estimated PDFs of the fitted distributions.

Turbocharger data

This data set contains the time to failure (10^3h) of turbocharger of a type of engine from Xu et al. (2003). These data were studied by Alzaatreh et al. (2016) and Cordeiro et al. (2019) using Weibull-gamma {log-logistic} and odd Lomax-Lomax distributions, respectively. For this data set, we fit E-L {GW}, EEL, PRHL, GG, and TEV models. The sample data is slightly left-skewed and slightly flatter than normal. It is anticipated that all distributions should fit properly. Table 5 shows all models fit the data set properly, while E-L {GW} has a better fit according to the p-values of the KS test statistics. As noticed, the shape parameter estimates of the four distributions that fit better to the data are not statistically significant. This is not surprising since the degree of left-skewness is minor. However, without shape parameter, symmetric distributions do not fit the data properly. Figure 8 shows the fitted models to the turbocharger data set.

Generalized logistic regression model applied to censored class-H insulation data

The data are hours to failure of 40 motorettes with a new Class-H insulation run at 190 °C, 220 °C, 240 °C, and 260 °C by Nelson (2004). Midway between the inspection time when the failure is found, and the time of the previous inspection is considered the failure time. The test aims to estimate the median life of such insulation at its design temperature of 180 °C. A median life of over 20,000 h is desired. The data consist

Table 4 MLEs, their standard errors (SEs) (in parentheses) and goodness of fit measures for the Adiponectin's data set

	MLE			$-\ell(\hat{\theta})$	AIC	KS (p-value)
E-L{GW}(μ, σ, ξ)	−3.3614 (9.6673)	−8.4609 (3.0491)	8.6885 (8.9183)	354.2757	714.5514	0.0447 (0.9746)
EEL(a, λ, θ)	3.6630 (0.6037)	0.2230 (0.0215)	1.1250 (0.0070)	357.1828	720.3655	0.0834 (0.3950)
PRHL(μ, λ, a)	−12.5762 (7.7934)	0.2316 (0.0179)	102.6242 (177.9828)	360.8723	727.7447	0.0829 (0.4019)
GG(μ, σ, β)	7.7414 (0.4331)	−2.6167 (0.3881)	− 0.5155 (0.0982)	356.2521	718.5042	0.0886 (0.3224)
TEV(μ, σ, a)	8.3690 (0.7213)	4.7540 (0.4903)	0.3981 (0.2281)	359.6354	725.2708	0.0717 (0.5908)

Fig. 7 Adiponectin data set: **a** Fitted E-L {GW}, GG and TEV PDFs. **b** Fitted E-L {GW}, GG and TEV CDFs
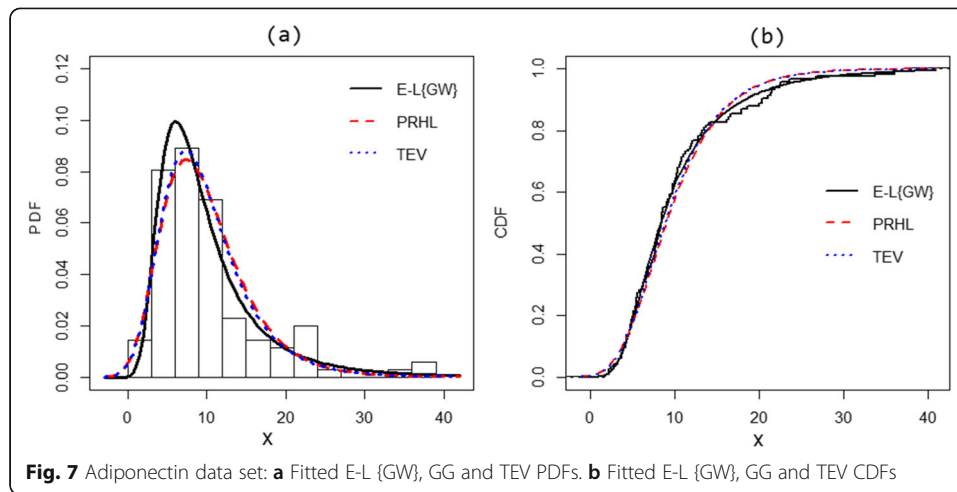
of ($n = 40$) observations (observed or right censored). The censoring indicator is 0 for censoring and 1 for observed. Each motorette is assigned one of the four test stress levels (10 motorettes in each level). Seven motorettes (1 in level 220, 1 in level 240, and 5 in level 260) are lost to follow-up and considered censored. The response variable $y_i = \log(t_i)$ is the logarithm of failure times (hours) t_i or the logarithm of the censoring time c_i, and the covariate v_i refers to the test stress levels (190, 220, 240, and 260).

The data are analyzed to determine the relationship between y and the level of test stress (v). The following regression model is considered:

$$y_i = \beta_0 + \beta_1 v_i^* + \sigma z_i,$$

where $v_i^* = (v_i - 180)$ is the centered stress level obtained by subtracting the design stress value 180, and y_i follows the E-L {GW} distribution in (21) with the shape parameter $\xi_i = \exp(\gamma_0 + \gamma_1 v_i^*)$ for $i = 1, ..., 40$. The model parameters in these applications are estimated by maximum likelihood method. Table 6 indicates that the AIC, AICC, and BIC statistic values of the E-L {GW} regression model are smaller than those of the other fitted models. The estimates β_1 and γ_1 are significant at the 5% level, and the levels of test stress have significant differences. The likelihood ratio (LR) statistic is used to compare the E-L {GW} regression model with some nested models. As shown in Table 6, the E-L {GW} model gives better fit to these data than the other nested models. Table 7 shows the LR statistics and the corresponding *p*-values. The

Table 5 MLEs, their SEs (in parentheses) and goodness of fit measures for the turbocharger data set

	MLE			$-\ell(\hat{\theta})$	AIC	KS (*p*-value)
E-L{GW}(μ, σ, ξ)	7.7777 (1.1696)	1.8229 (0.5912)	1.9295 (1.8198)	80.0798	166.1595	0.0741 (0.9807)
EEL(a, λ, θ)	161.0992 (725.8089)	4.0828 (6.0810)	5.8891 (6.8837)	87.1234	180.2468	0.1315 (0.4930)
PRHL(μ, λ, a)	8.6753 (0.3141)	0.1959 (0.1672)	0.0801 (0.0769)	80.4757	166.9513	0.1141 (0.6750)
GG(μ, σ, β)	−66.6650 (299.6159)	−199.3733 (817.4937)	−128.7707 (528.6058)	83.7637	173.5274	0.0899 (0.9038)
TEV(μ, σ, a)	4.5182 (0.4648)	1.9043 (0.2130)	−0.6244 (0.2855)	87.8765	181.7530	0.1421 (0.3942)

Fig. 8 Turbocharger data set: **a** Fitted E-L {GW}, GG and PRHL PDFs. **b** Fitted E-L {GW}, GG and PRHL CDFs
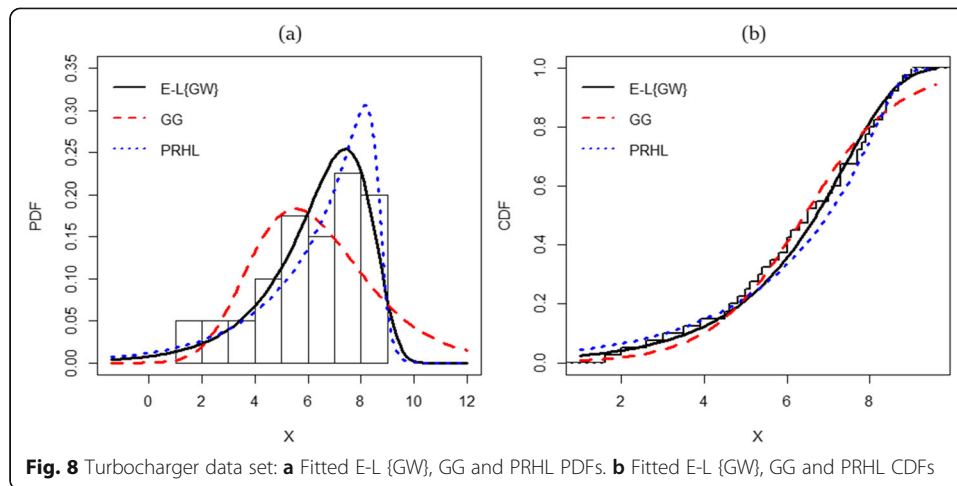
implication of the results in Table 7 is that the E-L {GW} outperforms all the sub-models. Thus, one should use the E-L {GW} regression model to analyze the data.

Generalized logistic regression model applied to censored heart transplant data

The data consist of $n = 103$ heart transplant patients of which 69 patients received transplants and 34 did not. The data were from Crowley and Hu (1977) and reported by Kalbfleisch and Prentice (2002). The data can be used to assess the effect of transplantation on patients' survival. The response variable $y_i = \log(t_i)$ is the logarithm of survival time in days (the time from the enrollment until death or until the study ended). The covariates are v_{i1} (age in years at acceptance) and v_{i2} (transplant status: 1 = transplanted, 0 = not transplanted). The survival status or censoring indicator is 0 for alive and 1 for dead. Thus, the data are analyzed to investigate the relationship between survival time and the covariates age and transplant status. The following regression model is considered:

$$y_i = \beta_0 + \beta_1 v_{i1} + \beta_2 v_{i2} + \sigma z_i,$$

Table 6 MLEs of the parameters (SE in parentheses), *p*-values bellow SE, and goodness of fit measures for the Class-H Insulation Data

Model	σ	β_0	β_1	γ_0	γ_1	$-\ell(\boldsymbol{\theta})$	AIC	AICC	BIC
E-L {GW}	−0.2459	8.5954	−0.0165	4.6284	−0.1272	5.3258	20.6516	22.4163	29.0960
	(0.0551)	(0.2845)	(0.0046)	(1.0436)	(0.0299)				
		<.0001	0.0008	<.0001	<.0001				
Logistic	0.1823	9.1328	−0.02463	–	–	14.1221	34.2442	34.9108	39.3108
	(0.0257)	(0.1024)	(0.0020)						
		<.0001	<.0001						
Weibull	0.2529	9.2304	−0.0239	–	–	13.8300	33.6600	34.3267	38.7266
	(0.0352)	(0.0739)	(0.0014)						
		<.0001	<.0001						
Fréchet	−0.3042	9.0075	−0.02501	–	–	14.5422	35.0845	35.7511	40.1511
	(0.0387)	(0.1418)	(0.0028)						
		<.0001	<.0001						

Table 7 LR statistics for the Class-H Insulation Data

Model	Hypotheses	LR statistic	p-value
E-L {GW} vs Logistic	$H_0 : \gamma_0 = -\infty, \gamma_1 = 0$	17.5925	0.0002
E-L {GW} vs Weibull	$H_0 : \gamma_0 = \gamma_1 = 0, \sigma > 0$	17.0084	0.0002
E-L {GW} vs Fréchet	$H_0 : \gamma_0 = \gamma_1 = 0, \sigma < 0$	18.4328	0.0001

where y_i follows the E-L {GW} distribution in (21) with the shape parameter $\xi_i = \exp(\gamma_0 + \gamma_1 v_{i1})$ for $i = 1, ..., 103$. The model parameters in these applications are estimated by maximum likelihood method. Table 8 indicates that the AIC, AICC, and BIC statistic values of the E-L {GW} regression model are smaller than those of the other fitted models. The estimates β_1, β_2, and γ_1 are significant at the 5% level, and the status of transplant have significant differences. The LR statistic is used to compare the E-L {GW} regression model with some nested models. Table 9 shows the LR statistics and the corresponding *p*-values. As shown in Table 8, the E-L {GW} model gives the best goodness of fit statistic among all models.

Summary and conclusions

The logistic and Gumbel (maximum and minimum) distributions have been widely studied, and many generalizations have been considered to model real-life applications. We propose a new generalization for the logistic and Gumbel distributions called the generalized exponential-logistic distribution. We study the structural properties of this new distribution and the relationships between the parameters and the mean, variance, skewness, and kurtosis. With only three parameters, the E-L {GW} can fit data with a very wide range of skewness (left and right) and kurtosis. The proposed method for developing generalized distributions has a high potential for practitioners. A generalized logistic regression model based on the E-L {GW} distribution is developed. Some existing regression models are sub-models, which makes the generalized logistic regression model a good choice for modeling a wide variety of response variables. Four real data sets are applied to illustrate the usefulness of the new distribution and its regression for fitting skewed data. The applications suggest that these generalized logistic and Gumbel distributions can fit highly skewed data sets effectively.

Table 8 MLEs of the parameters (SEs in parentheses), *p*-values bellow SE and goodness of fit measures for the Heart transplant data set

Model	σ	β_0	β_1	β_2	γ_0	γ_1	$-\ell(\boldsymbol{\theta})$	AIC	AICC	BIC
E-L {GW}	−1.4147 (0.2824) < 0.0001	7.2609 (1.3543) 0.0012	−0.1050 (0.0314) < 0.0001	2.9103 (0.3407) 0.0347	−5.7436 (2.6844) 0.0180	0.1108 (0.0461)	165.5161	343.0322	343.9072	358.8406
Logistic	1.0231 (0.0971) < 0.0001	6.0800 (0.9129) 0.0008	−0.0682 (0.0197) < 0.0001	2.8609 (0.3922)	–	–	170.4686	348.9372	349.3454	359.4761
Weibull	1.4897 (0.1348) < 0.0001	8.1193 (0.9845) < 0.0001	−0.0951 (0.0215) < 0.0001	2.8112 (0.3698)	–	–	173.9400	355.8800	356.5050	364.8164
Fréchet	−1.7496 (0.1501) < 0.0001	4.2179 (0.9094) 0.0361	−0.0398 (0.0187) < 0.0001	2.5461 (0.3695)	–	–	170.6174	349.2348	349.8598	359.7737

Table 9 LR statistics for the Heart transplant data set

Model	Hypotheses	LR statistic	*p*-value
E-L {GW} vs Logistic	$H_0 : \gamma_0 = -\infty, \gamma_1 = 0$	9.9050	0.0071
E-L {GW} vs Weibull	$H_0 : \gamma_0 = \gamma_1 = 0, \sigma > 0$	16.8478	0.0002
E-L {GW} vs Fréchet	$H_0 : \gamma_0 = \gamma_1 = 0, \sigma < 0$	10.2026	0.0061

Appendix

Proof of Theorem 1

The derivative of $f_X(x)$ in (7) is given by

$$f_X'(x) = -\frac{1}{\sigma |\sigma|} \left(\exp\left(\frac{x-\mu}{\sigma}\right) + 1 \right)^{\xi - 2} \exp\left(\left(\frac{x-\mu}{\sigma}\right) + \left\{ 1 - \left(\exp\left(\frac{x-\mu}{\sigma}\right) + 1 \right)^{\xi} - 1 \right\}/\xi \right) w(x),$$

where $w(x) = \exp(\frac{x-\mu}{\sigma})(\exp(\frac{x-\mu}{\sigma}) + 1)^{\xi} - \xi \exp(\frac{x-\mu}{\sigma}) - 1, \ -\infty < x < \infty$. By setting $w(x)$ to zero and replacing $\exp(\frac{x-\mu}{\sigma})$ by u, we obtain (10). If $\xi = \{0, 1\}$, then from (10) the mode is at $u = 1$, equivalently, $x = \mu$. When $\xi \neq \{0, 1\}$, then the curve on the right hand side of (10), $k(u) = u(u+1)^{\xi}$ is convex in u ($k''(u) > 0$ for all $u > 0$). Therefore, the curve $k(u)$ and the line $\xi u + 1$ on the left hand side of (10) can intersect at most twice. This means $w(x) = 0$ has at most two solutions, and so is $f'(x) = 0$. Now, since $\lim_{x \to -\infty} f_X(x) = \lim_{x \to \infty} f_X(x) = 0$, then $f_X(x)$ has exactly one mode. Note that if we assume that $f_X(x)$ has two modes (or more), then $w(x) = 0$ will have three solutions (two modes and local minimum). This is a contradiction with $w(x) = 0$ has at most two solutions, therefore, $f_X(x)$ is unimodal. □

Proof of Corollary 2

When $\sigma > 0$, the derivative of the hazard function in (9) is given by

$$h'(x) = \frac{1}{\sigma^2} \exp\left(\frac{x-\mu}{\sigma}\right) \left\{ \xi \exp\left(\frac{x-\mu}{\sigma}\right) + 1 \right\} \left\{ \exp\left(\frac{x-\mu}{\sigma}\right) + 1 \right\}^{\xi - 2}. \tag{23}$$

From (23), $h'(x) \geq 0$ for all $-\infty < x < \infty$, therefore, $h(x)$ is increasing whenever $\sigma > 0$. When $\sigma < 0$ and by using L'Hopital's rule, we find that

$$\lim_{x \to \infty} h(x) = \lim_{x \to \infty} \frac{\frac{1}{\sigma} \exp(\frac{x-\mu}{\sigma})\{ \exp(\frac{x-\mu}{\sigma}) + 1\}^{\xi - 1} + (\xi - 1) \exp(2\frac{x-\mu}{\sigma})\{ \exp(\frac{x-\mu}{\sigma}) + 1\}^{\xi - 2}}{\frac{1}{\sigma} |\sigma| \ \exp(\frac{x-\mu}{\sigma})\{ \exp(\frac{x-\mu}{\sigma}) + 1\}^{\xi - 1} \exp(\frac{1}{\xi}[\{ \exp(\frac{x-\mu}{\sigma}) + 1\}^{\xi} - 1])} = \frac{1}{|\sigma|}. \ \square$$

Proof of Theorem 2

Let $Z = (X - \mu)/\sigma$, and using binomial expansion, yields

$$E(|X|^n) \leq \sum_{i=0}^{n} \binom{n}{i} |\mu|^{n-i} |\sigma|^i E|Z|^i, \tag{24}$$

where Z is E-L {GW} random variable with $\mu = 0$ and $\sigma = 1$.

Now, using definition, we have

$$E(|Z|^i) = \int_{-\infty}^{\infty} |z|^i \exp(z)(1 + \exp(z))^{\xi - 1} \exp\left\{ -\left[(1 + \exp(z))^{\xi} - 1 \right]/\xi \right\} dz,$$

$$= \int_{-\infty}^{\infty} |z|^i \frac{\exp(z)}{(1+\exp(z))^2} g(z)dz, \tag{25}$$

where $g(z) = (1 + \exp(z))^{\xi+1} \exp\{-[(1 + \exp(z))^{\xi} - 1]/\xi\}$. By using the elementary calculus, we find that $\sup_{-\infty < z < \infty}\{g(z)\} = e^{-1}(1 + \xi)^{1/\xi+1}$. From (25) we obtain,

$$E(|Z|^i) \le e^{-1}(1+\xi)^{1/\xi+1} E(|L|^i), \tag{26}$$

where $E(|L|^i) = \int_{-\infty}^{\infty} |z|^i \frac{\exp(z)}{(1+\exp(z))^2}dz$ is the ith absolute moment of standard logistic distribution.

Using (26) in (24), the result in (13) is obtained. □

Proof of Theorem 3

Let $Z = (X - \mu)/\sigma$. We have

$$E(X^r) = \sum_{n=0}^{r} \binom{r}{n} \mu^{r-n} \sigma^n E(Z^n).$$

Using Eq. (11), the moments $E(Z^n)$ are obtained as

$$E(Z^n) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \omega_{i,j} E\left(L_{j+1}^n\right). \tag{27}$$

Therefore, the result in (14) is obtained from (27) directly. □

Proof of Proposition 3

Let $Z = (X - \mu)/\sigma$, then the MGF of Z can be written as

$$M_Z(t) = \int_{-\infty}^{\infty} \exp(zt + z)(1 + \exp(z))^{\xi-1} \exp\left(-\left[(1 + \exp(z))^{\xi} - 1\right]/\xi\right)dz. \tag{28}$$

On setting $u = [(1 + \exp(z))^{\xi} - 1]/\xi$ in (28), we obtain

$$M_Z(t) = \int_0^{\infty} \left((1 + \xi u)^{1/\xi} - 1\right)^t \exp(-u)du. \tag{29}$$

Using the generalized binomial theorem $(x + y)^\alpha = \sum_{i=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-i+1)\Gamma(i+1)} x^i y^{\alpha-i}, |x| < |y|$, (29) can be written as

$$M_Z(t) = \sum_{i=0}^{\infty} \frac{\Gamma(t+1)(-1)^i}{\Gamma(t-i+1)\Gamma(i+1)} \int_0^{\infty} (1 + \xi u)^{(t-i)/\xi} \exp(-u)du.$$

By using formula (3.382–4) in Gradshteyn and Ryzhik (2000), we obtain

$$M_Z(t) = \sum_{i=0}^{\infty} \frac{\Gamma(t+1)(-1)^i}{\Gamma(t-i+1)\Gamma(i+1)} \xi^{(t-i)/\xi} \exp(1/\xi)\Gamma((t-i)/\xi + 1, 1/\xi). \tag{30}$$

Now, the MGF of the $X = \mu + \sigma Z$ is defined as

$$M_X(t) = E(\exp(Xt)) = \exp(\mu t)M_Z(\sigma t). \tag{31}$$

Using (31) with (30), the result in (15) is obtained.

Note that the values of the augment t that makes (15) exist can be obtained directly from (29) by noting that $u < ((1 + \xi u)^{1/\xi} - 1) < e^u$ when $u > 0$, $0 < \xi < 1$, and $0 < ((1 + \xi u)^{1/\xi} - 1) < u$ when $u > 0$, $\xi \geq 1$. □

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