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Characterizations of folded student's *t* distribution

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Abstract

A probability distribution can be characterized through various methods. In this paper, we have established some new characterizations of folded Student's *t* distribution by truncated first moment, order statistics and upper record values. It is hoped that the results will be quite useful in the fields of probability, statistics, and other applied sciences.

Keywords: Characterization, Folded student's *t* distribution, Order statistics, Reverse hazard rate, Truncated first moment, Upper record values

2010 Mathematics Subject Classifications: 60E05, 62E10, 62E15, 62G30

1. Introduction

Characterization of a probability distribution plays an important role in probability and statistics. Before a particular probability distribution model is applied to fit the real world data, it is necessary to confirm whether the given continuous probability distribution satisfies the underlying requirements by its characterization. A probability distribution can be characterized through various methods (see, for example, Ahsanullah et al. (2014), among others). Since the characterizations of probability distributions play an important part in the determination of distributions by using certain properties in the given data, there has been a great interest, in recent years, in the characterizations of probability distributions by truncated moments. For example, the development of the general theory of the characterizations of probability distributions by truncated moment began with the work of Galambos and Kotz (1978). Further development continued with the contributions of many authors and researchers, among them Kotz and Shanbhag (1980), Glänzel et al. (1984), and Glänzel (1987) are notable. Most of these characterizations are based on a simple relationship between two different moments truncated from the left at the same point. As pointed out by Glänzel (1987), these characterizations may serve as a basis for parameter estimation. The characterizations by truncated moments may also be useful in developing some goodness-of-fit tests of distributions by using data whether they satisfy certain properties given in the characterizations of distributions. For example, as pointed out by Kim and Jeon (2013), in actuarial science, the credibility theory proposed by Bühlmann (1967) allows actuaries to estimate the conditional mean loss for a given risk to establish an adequate premium to cover the insured's loss. In their paper, Kim and Jeon (2013) have proposed a credibility theory based on truncation of the loss data, or the trimmed mean, which also contains the classical credibility theory of Bühlmann (1967) as a special case. It appears from the literature that not much attention has been paid to the characterization of the folded Student's t distribution with n degrees of freedom. For the folded Student's t distribution with $n = 2$ degrees of freedom, the interested readers are referred to Ahsanullah et al. (2014). In this paper, motivated by the importance of the Student's t distribution in many practical problems when only the magnitudes of deviations are recorded and the signs of the deviations are ignored, we have established some new characterizations of folded Student's t distribution by truncated first moment, order statistics and upper record values, which, we hope, will be useful for practitioners and researchers in the fields of probability, statistics, and other applied sciences, such as, actuarial science, economics, finance, among others.

The organization of this paper is as follows. Section 2 discusses briefly the folded Student's t distribution and some of its properties. The characterizations of the folded Student's t distribution are presented in section 3. The concluding remarks are provided in Section 4. We have provided two lemmas in Appendix A (as Lemma A.1 and Lemma A.2) to prove the main results of the paper.

2. Folded student's *t* distribution and its distributional properties

In this section, we briefly discuss the folded Student's t distribution and some of its distributional properties.

2.1 Folded student's *t* distribution

An important class of probability distributions, known as the folded distributions, arises in many practical problems when only the magnitudes of deviations are recorded, and the signs of the deviations are ignored. The folded Student's t distribution is one such probability distribution which belongs to this class. It is related to the Student's t distribution in the sense that if Y is a distributed random variable having Student's t distribution, then the random variable $X = |Y|$ is said to have a folded Student's t distribution. The distribution is called folded because the probability mass (that is, area) to the left of the point $x = 0$ is folded over by taking the absolute value. As pointed out above, such a case may be encountered if only the magnitude of some random variable is recorded, without taking into consideration its sign (that is, its direction). Further, this distribution is used when the measurement system produces only positive measurements, from a normally distributed process. The folded Student's t distribution was developed by Psarakis and Panaretoes (1990). For details on the folded Student's t distribution, see Johnson et al. (1994). For some bivariate extension of the folded Student's t distribution, the interested readers are referred to Psarakis and Panaretos (2001). Recently, many researchers have studied the statistical methods dealing with the properties and applications of the folded Student's t distribution, among them Brazauskas and Kleefeld (2011, 2014), and Scollnik (2014), are notable.

Definition: Let Y be a random variable having the Student's t distribution with n degrees of freedom. Let $X = |Y|$. Then X has a folded Student's t distribution with n degrees of freedom and its pdf $f_X(x)$ is given by

$$f_X(x) = \frac{2}{\sqrt{n}\ B\left(\frac{n}{2},\ \frac{1}{2}\right)}\left(1+\frac{x^2}{n}\right)^{\frac{-(n\ +\ 1)}{2}}, \quad x > 0, \tag{1}$$

where $B(.,\ .)$ denotes the beta function; see, for example, Gradshteyn and Ryzhik (1980), among others. To describe the shapes of the folded Student's t distribution, the plots of the pdf (1) for some values of the parameter n are provided in Fig. 1. The effects of the parameter can easily be seen from these graphs. Similar plots can be drawn for others values of the parameters.

2.2 Moment of folded student's *t* distribution

For $n = 1$, $E(X)$ and $E(X^2)$ do not exist for the folded Student's t distribution having the pdf (1). When $n > 1$, the mean, $E(X)$, the second moment, $E(X^2)$, and the variance, $Var\ (X)$, for the folded Student's t distribution having the pdf (1), are respectively given as follows (see, for example, Psarakis & Panaretoes (1990)):

$$E\ (X) = \begin{cases} 2\sqrt{\frac{n}{\pi}}\frac{\Gamma\left(\frac{n+1}{2}\right)}{(n\ -\ 1)\Gamma\left(\frac{n}{2}\right)}, & n > 1, \\ \infty, & n = 1 \end{cases} \tag{2}$$

$$E\left(X^2\right) = \frac{n}{n\ -\ 2}, \quad n > 2$$

and

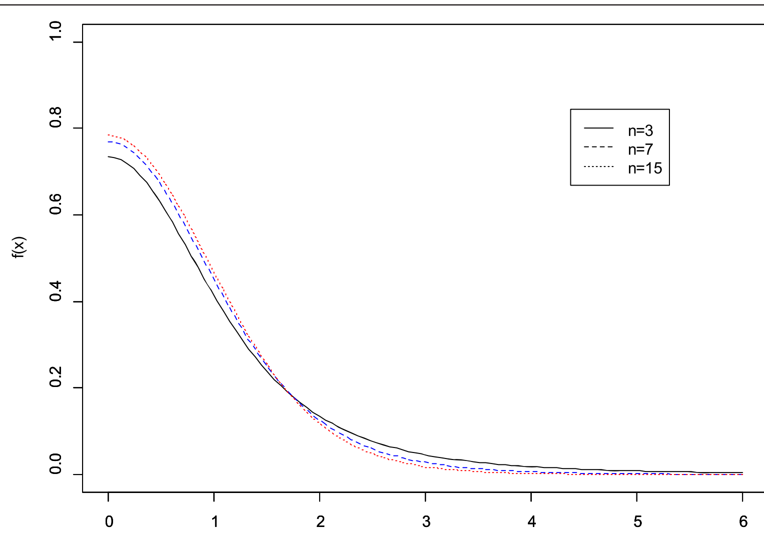

Fig. 1 Plots of the folded student's *t* distribution pdf

$$Var\ (X) = \begin{cases} \dfrac{n}{n-2} - \dfrac{4n}{\pi(n-1)^2}\left[\dfrac{\Gamma\left(\dfrac{n+1}{2}\right)}{\Gamma\left(\dfrac{n}{2}\right)}\right]^2, & n > 2, \\ \infty, & n \leq 2 \end{cases} \tag{3}$$

where Γ(.) denotes the gamma function. Now, noting the following well-known asymptotic relation

$$\lim_{x\to\infty}\left[\frac{\Gamma(x+\alpha)}{x^{\alpha}\Gamma(x)}\right] = 1$$

for real α and x, see, for example, Wendel (1948), and Abramowitz and Stegun (1970), page 257, 6.1.46, among others, and using this in the above expressions (2) and (3) for $E\ (X)$ and $Var\ (X)$ respectively, it can easily be seen that, in the limit, we have the following:

$$\lim_{n\to\infty}E(X) = \lim_{n\to\infty}\left[2\sqrt{\frac{n}{\pi}}\frac{\Gamma\left(\frac{n+1}{2}\right)}{(n-1)\Gamma\left(\frac{n}{2}\right)}\right] = \sqrt{\frac{2}{\pi}}\approx 0.79789,$$

and

$$\lim_{n\to\infty}Var(X) = \lim_{n\to\infty}\left\{\frac{n}{n-2} - \frac{4n}{\pi(n-1)^2}\left[\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\right]^2\right\} = 1-\frac{2}{\pi}$$
$$= \frac{\pi-2}{\pi}\approx 0.36338.$$

For more distributional properties of folded Student's t distribution, the interested readers are referred to Psarakis and Panaretoes (1990), and Johnson et al. (1994), among others.

3. Characterization of folded student's *t* distribution

In this section, we present some new characterizations of folded Student's t distribution by truncated first moment, order statistics and upper record values, as given below.

3.1 Characterization by truncated first moment

The characterizations of folded Student's t distribution by truncated first moment are provided below.

Assumption 3.1: Suppose the random variable X is absolutely continuous with the cumulative distribution function $F(x)$ and the probability density function $f(x)$. We assume that $\gamma = \inf\{x \mid F(x) > 0\}$, and $\delta = \sup\{x \mid F(x) < 1\}$. We also assume that $E(X)$ exists.

Theorem 3.1.1: If the random variable X satisfies the Assumption 3.1 with $\gamma = 0$ and $\delta = \infty$, then $E(X|X \leq x) = g(x)\,\tau(x)$, where $\tau(x) = \frac{f(x)}{F(x)}$ and $g(x) = \frac{n}{n-1}\left[\left(1+\frac{x^2}{n}\right)^{\frac{n+1}{2}} - \left(1+\frac{x^2}{n}\right)\right]$, if and only if X has the folded Student's t distribution with the pdf as given in Eq. (1).

Proof: Suppose that $E(X|X \leq x) = g(x)\,\tau(x)$. Then, since $E(X|X \leq x) = \dfrac{\int_0^x uf(u)\,du}{F(x)}$, and $\tau(x) = \frac{f(x)}{F(x)}$, we have $g(x) = \dfrac{\int_0^x uf(u)\,du}{f(x)}$. Now, if the random variable X satisfies the Assumption 3.1 and has the folded Student's t distribution as given in Eq. (1), then, after simplification, we have

$$g(x) = \frac{\int_0^x uf(u)\,du}{f(x)} = \left(1 + \frac{x^2}{n}\right)^{\frac{n+1}{2}} \int_0^x u\left(1 + \frac{u^2}{n}\right)^{-\frac{n+1}{2}} du$$
$$= \frac{n}{n-1}\left[\left(1 + \frac{x^2}{n}\right)^{\frac{n+1}{2}} - \left(1 + \frac{x^2}{n}\right)\right].$$

Conversely, suppose that $g(x) = \frac{n}{n-1}\left[\left(1 + \frac{x^2}{n}\right)^{\frac{n+1}{2}} - \left(1 + \frac{x^2}{n}\right)\right]$. Then, after differentiation, we have

$$g'(x) = \frac{n}{n-1}\left\{\frac{n+1}{n} x\left(1 + \frac{x^2}{n}\right)^{\frac{n-1}{2}} - \frac{2x}{n}\right\}.$$

Using the above expressions for $g(x)$ and $g'(x)$, after simplification, we obtain

$$\frac{x - g'(x)}{g(x)} = -\frac{(n+1)}{n} x\left(1 + \frac{x^2}{n}\right)^{-1}.$$

Consequently, by using Lemma A.1, we obtain

$$\frac{f'(x)}{f(x)} = -\frac{(n+1)}{n} x\left(1 + \frac{x^2}{n}\right)^{-1}.$$

On integrating the above equation with respect to x, we obtain $f(x) = c\left(1 + \frac{x^2}{n}\right)^{-\frac{n+1}{2}}$, where c is a constant to be determined.

On integrating the above equation with respect to x from $x = 0$ to $x = \infty$, and using the condition $\int_0^\infty f(x)\,dx = 1$, and noting the integral representation of beta function, that is, $B(u, v) = \int\limits_0^\infty \frac{t^{u-1}}{(1+t)^{u+v}} dt, \quad u > 0, \quad v > 0$, (see, Gradshteyn and Ryzhik (1980), Eq. 8.380.3, Page 948), we obtain $c = \frac{2}{\sqrt{n}B\left(\frac{n}{2}, \frac{1}{2}\right)}$, and thus $f_X(x) = \frac{2}{\sqrt{n}\ B\left(\frac{n}{2}, \frac{1}{2}\right)}\left(1 + \frac{x^2}{n}\right)^{\frac{-(n+1)}{2}}, \quad x > 0$, which is the pdf of folded Student's t distribution with n degrees of freedom. This completes the proof.

Special Case: By taking $n = 3$ in Theorem 3.1.1, it is easy to see that, for an absolutely continuous (with respect to Lebesgue measure) non-negative random variable X with cdf $F(x)$ and pdf $f(x)$, we have $E(X|X \le x) = g(x)\,\tau(x)$, where $g(x) = \frac{x^2(x^2+3)}{6}$, and $\tau(x)$ is the reversed hazard rate function given by $\tau(x) = \frac{f(x)}{F(x)}$, if and only if X has the folded Student's t_3 distribution with the pdf $f(x) = \frac{2}{\pi\sqrt{3}}\left(1 + \frac{x^2}{3}\right)^{-2}, \quad x > 0.$

Theorem 3.1.2: If the random variable X satisfies the Assumption 3.1 with $\gamma = 0$ and $\delta = \infty$, then $E(X|X \ge x) = \tilde{g}(x)r(x)$, where $r(x) = \frac{f(x)}{1 - F(x)}$ and $\tilde{g}(x) = \frac{n}{n-1}\left(1 + \frac{x^2}{n}\right)$, if and only if X has the folded Student's t distribution with the pdf as given in Eq. (1).

Proof: Suppose that $E(X|X \geq x) = \tilde{g}(x) r(x)$. Then, since $E(X|X \geq x) = \frac{\int_x^\infty uf(u)\,du}{1 - F(x)}$ and $r(x) = \frac{f(x)}{1 - F(x)}$, we have $\tilde{g}(x) = \frac{\int_x^\infty uf(u)\,du}{f(x)}$. Now, if the random variable X satisfies the Assumption 3.1 and has the folded Student's t distribution as given in Eq. (1), then, after simplification, we have

$$\begin{aligned}\tilde{g}(x) &= \frac{\int_x^\infty uf(u)\,du}{f(x)} = \left(1 + \frac{x^2}{n}\right)^{\frac{n+1}{2}} \int_x^\infty u\left(1 + \frac{u^2}{n}\right)^{-\frac{n+1}{2}} du \\ &= \frac{n}{n-1}\left(1 + \frac{x^2}{n}\right).\end{aligned}$$

Conversely, suppose that $\tilde{g}(x) = \frac{n}{n-1}\left(1 + \frac{x^2}{n}\right)$. Then, after differentiation, we have

$$(\tilde{g}(x))^{/} = \frac{2x}{n-1}.$$

Using the above expressions for $\tilde{g}(x)$ and $(\tilde{g}(x))^{/}$, after simplification, we obtain

$$\frac{-\left[x + (\tilde{g}(x))^{/}\right]}{\tilde{g}(x)} = -\frac{(n+1)x}{n\left(1 + \frac{x^2}{n}\right)}.$$

Consequently, by using Lemma A.2, we have

$$\frac{f^{/}(x)}{f(x)} = -\frac{(n+1)x}{n\left(1 + \frac{x^2}{n}\right)}.$$

On integrating the above equation with respect to x, we obtain $f(x) = c\left(1 + \frac{x^2}{n}\right)^{-\frac{n+1}{2}}$, where c is a constant to be determined.

On integrating the above equation with respect to x from $x = 0$ to $x = \infty$, and noting that $\int_0^\infty f(x)dx = 1$, and $B(u, v) = \int_0^\infty \frac{t^{u-1}}{(1+t)^{u+v}} dt$, $u > 0$, $v > 0$, we obtain $c = \frac{2}{\sqrt{n}\,B\left(\frac{n}{2}, \frac{1}{2}\right)}$, and thus $f_X(x) = \frac{2}{\sqrt{n}\;B\left(\frac{n}{2}, \frac{1}{2}\right)}\left(1 + \frac{x^2}{n}\right)^{\frac{-(n+1)}{2}}$, $x > 0$, which is the pdf of folded Student's t distribution with n degrees of freedom. This completes the proof.

Note: It is interesting to point out here that there is connection between Theorem 3.1.1 and Theorem 3.1.2. For example, since, as in Theorem 3.1.1 and Theorem 3.1.2, $g(x) = \frac{\int_0^x uf(u)\,du}{f(x)}$ and $\tilde{g}(x) = \frac{\int_x^\infty uf(u)\,du}{f(x)}$ respectively, we have $g(x) + \tilde{g}(x) = \frac{\int_0^\infty uf(u)\,du}{f(x)} = \frac{E(X)}{f(x)}$. that is,

$$g(x) = \frac{E(X)}{f(x)} - \tilde{g}(x), \text{ or } \tilde{g}(x) = \frac{E(X)}{f(x)} - g(x),$$

where $E(X)$ denotes the mean, as given in Eq. (2), of folded Student's t distribution with the pdf (1). Thus, using the expressions for $f(x)$ as in Eq. (1), and for $E(X)$ as in Eq. (2), respectively, we can easily derive the formulas for $g(x)$ or $\tilde{g}(x)$ when X follows the folded Student's t distribution.

3.2 Characterization by order statistics

The characterizations of folded Student's t distribution by order statistics are provided in Theorem 3.2.1 and Theorem 3.2.2 below. We will consider the pdf $f(x)$ of the folded Student's t distribution as given in Eq. (1). Let $X_1, X_2, \ldots, X_n$ be n independent copies of the random variable X having absolutely continuous distribution function $F(x)$ and pdf $f(x)$. Suppose that $X_{1,n} \le X_{2,n} \le \ldots \le X_{n,n}$ are the corresponding order statistics. It is known that $X_{j,n}|X_{k,n} = x$, for $1 \le k < j \le n$, is distributed as the $(j-k)th$ order statistics from $(n-k)$ independent observations from the random variable V having the pdf $f_V(v|x)$ where $f_V(v|x) = \frac{f(v)}{1-F(x)}$, $0 \le v < x$, see, for example, Ahsanullah et al. (2013), Chapter 5, or Arnold et al. (2005), Chapter 2, among others. Further, $X_{i.,n}|X_{k,n} = x$, $1 \le i < k \le n$, is distributed as ith order statistics from k independent observations from the random variable W having the pdf $f_W(w|x)$ where $f_W(w|x) = \frac{f(w)}{F(x)}$, $w < x$. We assume that $S_{k-1} = \frac{1}{k-1}\left(X_{1,n} + X_{2,n} + \ldots + X_{k-1,n}\right)$, and $T_{k,n} = \frac{1}{n-k}\left(X_{k+1,n} + X_{k+2,n} + \ldots + X_{n.n}\right)$.

Theorem 3.2.1: Suppose the random variable X satisfies the Assumption 3.1 with $\gamma = 0$ and $\delta = \infty$, then $E(S_{k-1}|X_{k,n} = x) = g(x)\,\tau(x)$, where $g(x) = \frac{n}{n-1}\left[\left(1+\frac{x^2}{n}\right)^{\frac{n+1}{2}} - \left(1+\frac{x^2}{n}\right)\right]$ and $\tau(x) = \frac{f(x)}{F(x)}$, if and only if X has the folded Student's t distribution with the pdf as given in Eq. (1).

Proof: Since $E\left(S_{k-1}|X_{k,n} = x\right) = \frac{\int_0^x uf(u)\,du}{F(x)} = E(X|X \le x)$, the proof of Theorem 3.2.1 follows from Theorem 3.1.1.

Theorem 3.2.2: Suppose the random variable X satisfies the Assumption 3.1 with $\gamma = 0$ and $\delta = \infty$, then $E\left(T_{k,n}|X_{k,n} = x\right) = \tilde{g}(x)r(x)$, where $r(x) = \frac{f(x)}{1-F(x)}$ and $\tilde{g}(x) = \frac{n}{n-1}\left(1+\frac{x^2}{n}\right)$, if and only if X has the folded Student's t distribution with the pdf as given in Eq. (1).

Proof: Since $E\left(T_{k,n}|X_{k,n} = x\right) = \frac{\int_x^\infty uf(u)\,du}{1-F(x)} = E(X|X \ge x)$, the proof follows from Theorem 3.1.2.

3.3 Characterization by upper record values

In this section, we will establish a theorem on the characterization of the folded Student's t distribution by upper record values. Suppose that $X_1, X_2, \ldots$ is a sequence of independent and identically distributed absolutely continuous random variables with distribution function $F(x)$ and pdf $f(x)$. Let $Y_n = \max(X_1, X_2, \ldots, X_n)$ for $n \ge 1$. We say that X_j is an upper record value of $\{X_n, n \ge 1\}$ if $Y_j > Y_{j-1}, j > 1$. The indices at which the upper records occur are given by the record times $\{U(n) > \min(j \mid j >$

$U(n-1), X_j > X_{U(n-1)}, n > 1)\}$ and $U(1) = 1$. We will denote the *nth* upper record value as $X(n) = X_{U(n)}$. The pdf of $X(n)$ is given by

$$f_n(x) = \frac{(R(x))^{n-1}}{\Gamma(n)} f(x), \quad -\infty < x < \infty,$$

where $R(x) = -\ln(1-F(x))$ denotes the cumulative hazard rate function. The joint pdf $f_{n,\,n+1}(x,\,y)$ of $X(n)$ and $X(n+1)$ is given by (see Ahsanullah (1995), page 4)

$$f_{n,\,n+1}(x,\,y) = \frac{(R(x))^{n-1}}{\Gamma(n)} r(x) f(y), \quad -\infty < x < y < \infty,$$

where $r(x) = \frac{dR(x)}{dx} = \frac{f(x)}{1-F(x)}$ denotes the hazard rate function. Thus, the conditional pdf $f_{n+1,\,n}(y \mid x)$ of $X(n+1)|X(n)=x$ is $f_{n+1,\,n}(y \mid x) = \frac{f_{n,\,n+1}(x,\,y)}{f_n(x)} = \frac{f(y)}{1-F(x)}$. Based on these results, we now have the characterization of folded Student's t distribution by upper record values, as provided in Theorem 3.3.1 below. We will consider the pdf $f(x)$ of the folded Student's t distribution as given in Eq. (1).

Theorem 3.3.1: Suppose the random variable X satisfies the Assumption 3.1 with $\gamma = 0$ and $\delta = \infty$. Then X has the folded Student's t distribution with n degrees of freedom if and only if

$E(X(n+1)|X(n)=x) = \tilde{g}(x) r(x)$, where $r(x) = \frac{f(x)}{1-F(x)}$ and $\tilde{g}(x) = \frac{n}{n-1}\left(1 + \frac{x^2}{n}\right)$.

Proof: Since, as shown above, $f_{n+1,\,n}(y \mid x) = \frac{f(y)}{1-F(x)}$, it follows that

$$E(X(n+1)\,|X(n)=x) = \frac{\int_x^\infty u f(u)\,du}{1-F(x)} = E(X|X \geq x).$$

And, hence, the proof of Theorem 3.3.1 follows from Theorem 3.1.2.

4. Concluding remarks

Characterization of a probability distribution plays an important role in probability and statistics, and other applied sciences. Before a particular probability distribution model is applied to fit the real world data, it is necessary to confirm whether the given probability distribution satisfies the underlying requirements by its characterization. A probability distribution can be characterized through various methods. Since the characterizations of probability distributions by truncated moments play an important part in the determination of distributions by using certain properties in the given data, in this paper we have established some new characterizations of folded Student's t distribution by truncated first moment, order statistics and upper record values. It is hoped that the findings of the paper will be useful for researchers in the fields of probability, statistics, and other applied sciences.

Appendix A

Here, we establish the following two lemmas (Lemma A.1 and Lemma A.2) which have been useful in proving our main results in Section 3.

Lemma A.1

Under the Assumption 3.1, if $E(X \mid X \le x) = g(x)\,\tau(x)$, where $\tau(x) = \frac{f(x)}{F(x)}$ and $g(x)$ is a continuous differentiable function of x with the condition that $\int_0^x \frac{u - g^{/}(u)}{g(u)}\,du$ is finite for $x > 0$, then $f(x) = c e^{\int_0^x \frac{u - g^{/}(u)}{g(u)}\,du}$, where c is a constant determined by the condition $\int_0^\infty f(x)dx = 1$.

Proof: Suppose that $E(X|X \le x) = g(x)\,\tau(x)$. Then, since $E(X|X \le x) = \frac{\int_0^x u f(u)\,du}{F(x)}$ and $\tau(x) = \frac{f(x)}{F(x)}$, we have $g(x) = \frac{\int_0^x u f(u)\,du}{f(x)}$, that is, $\int_0^x u f(u)\,du = f(x)g(x)$.

Differentiating both sides of the above equation with respect to x, we obtain

$$x f(x) = f^{/}(x) g(x) + f(x) g^{/}(x).$$

From the above equation, we obtain

$$\frac{f^{/}(x)}{f(x)} = \frac{x - g^{/}(x)}{g(x)}.$$

On integrating the above equation with respect to x, we have

$$f(x) = c e^{\int_0^x \frac{u - g^{/}(u)}{g(u)}\,du},$$

where c is obtained by the condition $\int_0^\infty f(x)dx = 1$. This completes the proof of Lemma A.1.

Lemma A.2

Under the Assumption 3.1, if $E(X \mid X \ge x) = \tilde{g}(x) r(x)$, where $r(x) = \frac{f(x)}{1 - F(x)}$ and $\tilde{g}(x)$ is a continuous differentiable function of x with the condition that $\int_x^\infty \frac{u + [\tilde{g}(u)]^{/}}{\tilde{g}(u)}\,du$ is finite for $x > 0$, then $f(x) = c e^{-\int_0^x \frac{u + [\tilde{g}(u)]^{/}}{\tilde{g}(u)}\,du}$, where c is a constant determined by the condition $\int_0^\infty f(x)dx = 1$.

Proof: Suppose that $E(X \mid X \ge x) = \tilde{g}(x) r(x)$. Then, since $E(X \mid X \ge x) = \frac{\int_x^\infty u f(u)\,du}{1 - F(x)}$ and $r(x) = \frac{f(x)}{1 - F(x)}$, we have $\tilde{g}(x) = \frac{\int_x^\infty u f(u)\,du}{f(x)}$, that is, $\int_x^\infty u f(u)\,du = f(x)\tilde{g}(x)$.

Differentiating the above equation with respect to respect to x, we obtain

$$-xf(x) = f^{/}(x)\tilde{g}(x) + f(x)[\tilde{g}(x)]^{/}.$$

From the above equation, we obtain

$$\frac{f^{/}(x)}{f(x)} = -\frac{x + [\tilde{g}(x)]^{/}}{\tilde{g}(x)}.$$

On integrating the above equation with respect to x, we have

$$f(x) = c e^{-\int_0^x \frac{u + [\tilde{g}(u)]^{/}}{\tilde{g}(u)} du},$$

where c is obtained by the condition $\int_0^\infty f(x)dx = 1$. This completes the proof of Lemma A.2.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
MA contributed to the properties and characterizations of folded student's *t* distribution, and participated in the draft of the manuscript. MS contributed to the studies of properties, graph and characterizations of folded student's distribution, drafted the manuscript, and participated in the sequence alignment. GK contributed to the properties and graph of folded student's t distribution, and participated in the draft of the manuscript. All authors read and approved the final manuscript.

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Declarations
We confirm that we have read Springer Open's guidance on competing interests and have included a statement indicating that none of the authors have any competing interests in the manuscript.

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