RESEARCH   **Open Access**

CrossMark

Skewness-kurtosis adjusted confidence estimators and significance tests

Wolf-Dieter Richter

Correspondence:
wolf-dieter.richter@uni-rostock.de
Institute of Mathematics, University
of Rostock, Ulmenstraße 69, Haus 3,
18057 Rostock, Germany

Abstract

First and second kind modifications of usual confidence intervals for estimating the expectation and of usual local alternative parameter choices are introduced in a way such that the asymptotic behavior of the true non-covering probabilities and the covering probabilities under the modified local non-true parameter assumption can be asymptotically exactly controlled. The orders of convergence to zero of both types of probabilities are assumed to be suitably bounded below according to an Osipov-type condition and the sample distribution is assumed to satisfy a corresponding tail condition due to Linnik. Analogous considerations are presented for the power function when testing a hypothesis concerning the expectation both under the assumption of a true hypothesis as well as under a modified local alternative. A limit theorem for large deviations by S.V. Nagajev/V.V. Petrov applies to prove the results. Applications are given for exponential families.

Keywords: Orders of confidence, Orders of modified local alternatives, True non-covering probabilities, Local non-true parameter choice, Covering probabilities, Linnik condition, Osipov-type condition, Skewness-kurtosis adjusted decisions, Order of significance, Error probabilities of first and second kind, Exponential family, Large deviations

Mathematics Subject Classification: 62E20; 62F05; 62F12; 60F10

1 Introduction

Asymptotic normality of the distribution of the suitably centered and normalized arithmetic mean of i.i.d. random variables is one of the best studied and most often exploited facts in asymptotic statistics. It is supplemented in local asymptotic normality theory by limit theorems for the corresponding distributions under the assumption that the mean is shifted of order $n^{-1/2}$. There are many successful simulations and real applications of both types of central limit theorems, and one may ask for a more detailed explanation of this success. The present note is aimed to present such additional theoretical explanation under certain circumstances. Moreover, the note is aimed to stimulate both analogous consideration in more general situations and checking the new results by simulation. Furthermore, based upon the results presented here, it might become attractive to search for additional explanation to various known simulation results in the area of asymptotic normality which is, however, behind the scope of the present note.

Based upon Nagaev's and Petrov's large deviation result in (Nagaev 1965; Petrov 1968), skewness-kurtosis modifications of usual confidence intervals for estimating the

expectation and of usual local alternative parameter choices are introduced here in a way such that the asymptotic behavior of the true non-covering probabilities and the covering probabilities under the modified local non-true parameter assumption can be exactly controlled. The orders of convergence to zero of both types of probabilities are suitably bounded below by assuming an Osipov-type condition, see (Osipov 1975), and the sample distribution is assumed to satisfy a corresponding Linnik condition, see (Ibragimov and Linnik 1971; Linnik 1961).

Analogous considerations are presented for the power function when testing a hypothesis concerning the expectation both under the assumption of a true hypothesis and under a local alternative. Finally, applications are given for exponential families.

A concrete situation where the results of this paper apply is the case sensitive preparing of the machine settings of a machine tool. In this case, second and higher order moments of the manipulated variable do not change from one adjustment to another one and may be considered to be known over time.

It might be another aspect of stimulating further research if one asks for the derivation of limit theorems in the future being close to those in (Nagaev 1965; Petrov 1968) but where higher order moments are estimated.

Let $X_1, \ldots, X_n$ be i.i.d. random variables with the common distribution law from a shift family of distributions, $P_\mu(A) = P(A - \mu), A \in \mathscr{B}$, where $\mathscr{B}$ denotes the Borel σ-field on the real line, the expectation equals $\mu, \mu \in R$, and the variance is σ^2. It is well known that $T_n = \sqrt{n}(\bar{X}_n - \mu)/\sigma$ is asymptotically standard normally distributed, $T_n \sim AN(0, 1)$. Hence, $P_\mu(T_n > z_{1-\alpha}) \to \alpha$, and under the local non-true parameter assumption, $\mu_{1,n} = \mu + \frac{\sigma}{\sqrt{n}}(z_{1-\alpha} - z_\beta)$, i.e. if one assumes that a sample is drawn with a shift of location (or with an error in the variable), then $P_{\mu_{1,n}}(T_n \leq z_{1-\alpha}) = P_{\mu_{1,n}}\left(\sqrt{n}\frac{\bar{X}_n - \mu_{1,n}}{\sigma} \leq z_\beta\right) \to \beta$ as $n \to \infty$, where z_q denotes the quantile of order q of the standard Gaussian distribution.

Let $ACI^u = \left[\bar{X}_n - \frac{\sigma}{\sqrt{n}}z_{1-\alpha}, \infty\right)$ denote the upper asymptotic confidence interval for μ where the true non-covering probabilities satisfy the asymptotic relation

$$P_\mu(ACI^u \text{ does not cover } \mu) \to \alpha, \ n \to \infty.$$

Because $P_{\mu_{1,n}}\left(\bar{X}_n - \frac{\sigma}{\sqrt{n}}z_{1-\alpha} < \mu\right) = P_{\mu_{1,n}}\left(\sqrt{n}\frac{\bar{X}_n - \mu_{1,n}}{\sigma} \leq z_\beta\right)$, the covering probabilities under $n^{-1/2}$-locally chosen non-true parameters satisfy

$$P_{\mu_{1,n}}(ACI^u \text{ covers } \mu) \to \beta, \ n \to \infty.$$

The aim of this note is to prove refinements of the latter two asymptotic relations where $\alpha = \alpha(n) \to 0$ and $\beta = \beta(n) \to 0$ as $n \to \infty$, and to prove similar results for two-sided confidence intervals and for the power function when testing corresponding hypotheses.

2 Expectation estimation

2.1 First and second kind adjusted one-sided confidence intervals

According to (Ibragimov and Linnik 1971; Linnik 1961), it is said that a random variable X satisfies the Linnik condition of order $\gamma, 0 < \gamma < 1/2$, if

$$E_\mu \exp\left\{|X - \mu|^{\frac{4\gamma}{2\gamma+1}}\right\} < \infty. \tag{1}$$

Let us define the first kind (or first order) adjusted asymptotic Gaussian quantile by

$$z_{1-\alpha(n)}(1) = z_{1-\alpha(n)} + \frac{g_1}{6\sqrt{n}}z_{1-\alpha(n)}^2$$

where $g_1 = E(X - E(X))^3/\sigma^{3/2}$ is the skewness of X. Moreover, let the first kind (order) adjusted upper asymptotic confidence interval for μ be defined by

$$ACI^u(1) = \left[\bar{X}_n - \frac{\sigma}{\sqrt{n}} z_{1-\alpha(n)}(1), \infty \right)$$

and denote a first kind modified non-true local parameter choice by

$$\mu_{1,n}(1) = \mu_{1,n} + \frac{\sigma g_1}{6n} \left(z_{1-\alpha(n)}^2 - z_{\beta(n)}^2 \right).$$

Let us say that the probabilities $\alpha(n)$ and $\beta(n)$ satisfy an Osipov-type condition of order γ if

$$n^\gamma \exp\left\{ \frac{n^{2\gamma}}{2} \right\} \cdot \min\{\alpha(n), \beta(n)\} \to \infty, \ n \to \infty. \tag{2}$$

This condition means that neither $\alpha(n)$ nor $\beta(n)$ tend to zero as fast as or even faster than $n^{-\gamma} \exp\{-n^{2\gamma}/2\}$, i.e. $\min\{\alpha(n), \beta(n)\} \gg n^{-\gamma} \exp\{-n^{2\gamma}/2\}$, and that $\max\{z_{1-\alpha(n)}, z_{1-\beta(n)}\} = o(n^\gamma), n \to \infty$. Here, $o(.)$ stands for the small Landau symbol.

If two functions f, g satisfy the relation $\lim_{n \to \infty} f(n)/g(n) = 1$ then this asymptotic equivalence will be expressed as $f(n) \sim g(n), n \to \infty$.

Theorem 1. *If $\alpha(n) \downarrow 0$, $\beta(n) \downarrow 0$ as $n \to \infty$ and conditions (1) and (2) are satisfied for $\gamma \in \left(\frac{1}{6}, \frac{1}{4} \right]$ then*

$$P_\mu(ACI^u(1) \text{ does not cover } \mu) \sim \alpha(n), \ n \to \infty$$

and

$$P_{\mu_{1,n}(1)}(ACI^u(1) \text{ covers } \mu) \sim \beta(n), \ n \to \infty.$$

Let us define the second kind adjusted asymptotic Gaussian quantile

$$z_{1-\alpha(n)}(2) = z_{1-\alpha(n)}(1) + \frac{3g_2 - 4g_1^2}{72n} z_{1-\alpha(n)}^3$$

where $g_2 = E(X - E(X))^4/\sigma^4 - 3$ is the kurtosis of X, the second kind adjusted upper asymptotic confidence interval for μ

$$ACI^u(2) = \left[\bar{X}_n - \frac{\sigma}{\sqrt{n}} z_{1-\alpha(n)}(2), \infty \right),$$

and a second kind modified non-true local parameter choice

$$\mu_{1,n}(2) = \mu_{1,n}(1) + \frac{\sigma \left(3g_2 - 4g_1^2 \right)}{72n^{3/2}} \left(z_{1-\alpha(n)}^3 - z_{\beta(n)}^3 \right).$$

Theorem 2. *If $\alpha(n) \downarrow 0$, $\beta(n) \downarrow 0$ as $n \to \infty$ and conditions (1) and (2) are satisfied for $\gamma \in \left(\frac{1}{4}, \frac{3}{10} \right]$ then*

$$P_\mu(ACI^u(2) \text{ does not cover } \mu) \sim \alpha(n), \ n \to \infty$$

and

$$P_{\mu_{1,n}(2)}(ACI^u(2) \text{ covers } \mu) \sim \beta(n), \ n \to \infty.$$

Remark 1. Under the same assumptions, analogous results are true for lower asymptotic confidence intervals, i.e. for $ACI^l(s) = \left(-\infty, \bar{X}_n + \frac{\sigma}{\sqrt{n}} z_{1-\alpha}^-(s) \right), s = 1, 2$:

$$P_\mu(ACI^l(s) \text{ does not cover } \mu) \sim \alpha(n)$$

and

$$P_{\mu_{1,n}^-(s)}(ACI^l(s) \text{ covers } \mu) \sim \beta(n), \, n \to \infty.$$

Here, $z_{1-\alpha}^-(s)$ means the quantity $z_{1-\alpha}(s)$ where g_1 is replaced by $-g_1, s = 1, 2$, and

$$\mu_{1,n}^-(s) = \mu - \frac{\sigma}{\sqrt{n}}(z_{1-\alpha} - z_\beta) + \frac{\sigma g_1}{6n}\left(z_{1-\alpha}^2 - z_\beta^2\right) - \frac{\sigma\left(3g_2 - 4g_1^2\right)}{72n^{3/2}}\left(z_{1-\alpha}^3 - z_\beta^3\right)I_{\{2\}}(s).$$

Remark 2. In many situations where limit theorems are considered as they were in Section 1, the additional assumptions (1) and (2) may, possibly unnoticed, be fulfilled. In such situations, Theorems 1 and 2, together with the following theorem, give more insight into the asymptotic relations stated in Section 1.

Theorem 3. *Large Gaussian quantiles satisfy the asymptotic representation*

$$z_{1-\alpha} = \sqrt{-2\ln\alpha - \ln|\ln\alpha| - \ln(4\pi)} \cdot \left(1 + O\left(\frac{\ln|\ln\alpha|}{(\ln\alpha)^2}\right)\right), \alpha \to +0.$$

Note that $O(.)$ means the big Landau symbol.

2.2 Two-sided confidence intervals

For $s \in \{1, 2\}, \alpha > 0$, put $L(s; \alpha) = \bar{X}_n - \frac{\sigma}{\sqrt{n}}z_{1-\alpha}(s)$ and $R(s; \alpha) = \bar{X}_n + \frac{\sigma}{\sqrt{n}}z_{1-\alpha}^-(s)$. Further, let $\alpha_i(n) > 0, i = 1, 2, \alpha_1(n) + \alpha_2(n) < 1$, and

$$ACI(s; \alpha_1(n), \alpha_2(n)) = [L(s; \alpha_1(n)), R(s; \alpha_2(n))].$$

If conditions (1) and (2) are fulfilled then $P_\mu((-\infty, L(s; \alpha_1(n)))\text{ covers }\mu) \sim \alpha_1(n)$ and $P_\mu((R(s; \alpha_2(n)), \infty)\text{ covers }\mu) \sim \alpha_2(n)$ as $n \to \infty$.

With more detailed notation $\mu_{1,n}(s) = \mu_{1,n}(s; \alpha, \beta)$ and $\mu_{1,n}^-(s) = \mu_{1,n}^-(s; \alpha, \beta)$,

$$P_{\mu_{1,n}(s; \alpha_1(n), \beta_1(n))}((L(s; \alpha_1(n)), \infty)\text{ covers }\mu) \sim \beta_1(n),$$
$$P_{\mu_{1,n}^-(s; \alpha_2(n), \beta_2(n))}((-\infty, R(s; \alpha_2(n)))\text{ covers }\mu) \sim \beta_2(n), \, n \to \infty.$$

The following corollary has thus been proved.

Corollary 1. *If $\alpha_1(n) \downarrow 0, \alpha_2(n) \downarrow 0$ as $n \to \infty$ and conditions (1) and (2) are satisfied for $\gamma \in \left(\frac{1}{6}, \frac{1}{4}\right]$ if $s = 1$ and for $\gamma \in \left(\frac{1}{4}, \frac{3}{10}\right]$ if $s = 2$, and with $(\alpha(n), \beta(n)) = (\alpha_1(n), \alpha_2(n))$, then*

$$P_\mu(ACI(s; \alpha_1(n), \alpha_2(n)) \text{ does not cover } \mu) \sim (\alpha_1(n) + \alpha_2(n)), \, n \to \infty.$$

Moreover,

$$\max_{\nu \in \{\mu_{1,n}(s; \alpha_1(n), \beta_1(n)), \mu_{1,n}^-(s; \alpha_2(n), \beta_2(n))\}} P_\nu(ACI(s) \text{ covers } \mu) \leq \max\{\beta_1(n), \beta_2(n)\}.$$

3 Testing

3.1 Adjusted quantiles

Let us consider the problem of testing the hypothesis $H_0 : \mu \leq \mu_0$ versus the alternative $H_A : \mu > \mu_0$. The first and second kind adjusted decision rules of the one-sided asymptotic Gauss test suggest to reject H_0 if $T_{n,0} > z_{1-\alpha(n)}(s)$ for $s = 1$ or $s = 2$, respectively, where $T_{n,0} = \sqrt{n}(\bar{X}_n - \mu_0)/\sigma$. Because

$$P_{\mu_0}(\text{reject } H_0) = P_{\mu_0}(ACI^u(s) \text{ does not cover } \mu_0),$$

it follows from Theorems 1 and 2 that under the conditions given there the (sequence of) probabilities of an error of first kind satisfy the asymptotic relation

$$P_{\mu_0}(reject\ H_0) \sim \alpha(n), n \to \infty.$$

Concerning the power function of this test, because

$$P_{\mu_{1,n}(s)}(do\ not\ reject\ H_0) = P_{\mu_{1,n}(s)}(ACI^u(s)\ covers\ \mu_0),$$

it follows under the same assumptions that the probabilities of a second kind error in the case that the sequence of the modified local parameters is $(\mu_{1,n}(s))_{n=1,2,\ldots}$, satisfy

$$P_{\mu_{1,n}(s)}(do\ not\ reject\ H_0) \sim \beta(n), n \to \infty.$$

Similar consequences for testing $H_1 : \mu > \mu_0$ or $H_2 : \mu \neq \mu_0$ are omitted, here.

3.2 Adjusted statistics

Let $T_n(1) = T_n - \frac{g_1}{6\sqrt{n}}T_n^2$ and $T_n(2) = T_n(1) - \frac{3g_2 - 8g_1^2}{72n}T_n^3$ be the first and second kind adjusted asymptotically Gaussian statistics, respectively, where $T_n = \frac{\sqrt{n}}{\sigma}(\bar{X}_n - \mu)$.

Theorem 4. *If the conditions* (1) *and* (2) *are satisfied for a certain* $\gamma \in \left(\frac{s}{2s+4}, \frac{s+1}{2s+6}\right]$ *where* $s \in \{1, 2\}$ *then*

$$P_{\mu_0}\left(T_n(s) > z_{1-\alpha(n)}\right) \sim \alpha(n), \ n \to \infty$$

and

$$P_{\mu_{1,n}(s)}\left(T_n(s) \leq z_{1-\alpha(n)}\right) \sim \beta(n), \ n \to \infty.$$

Clearly, the results of this theorem apply to both hypothesis testing and confidence estimation in a similar way as described in the preceding sections.

The material of the present paper is part of a talk presented by the author at the Conference of European Statistics Stakeholders, Rome 2014, see Abstracts of Communication, p.90, and arXiv:1504.02553. A more advanced 'testing-part' of this talk is presented in (Richter 2016) and deals with higher order comparisons of statistical tests.

4 Application to exponential families

Let ν denote a σ-finite measure and assume that the distribution P_ϑ has the Radon-Nikodym density $\frac{dP_\vartheta}{d\nu}(x) = \frac{e^{\vartheta x}}{\int e^{\vartheta x}\nu(dx)} = e^{\vartheta x - B(\vartheta)}$, say. For basics on exponential families we refer to Brown (1986). We assume that $X(\vartheta) \sim P_\vartheta$ and $X_1 = X(\vartheta) - EX(\vartheta) + \mu \sim \widetilde{P}_\mu$ where ϑ is known and μ is unknown. In the product-shift-experiment $\left[R^n, \mathscr{B}^n, \{\widetilde{P}_\mu^{\times n}, \mu \in R\}\right]$, expectation estimation and testing may be done as in Sections 2 and 3, respectively, where $g_1 = B'''(\vartheta)/(B''(\vartheta))^{3/2}$ and g_2 allows a similar representation.

Another problem which can be dealt with is to test the hypothesis $H_0 : \vartheta \leq \vartheta_0$ versus the alternative $H_{1n} : \vartheta \geq \vartheta_{1n}$ if one assumes that the expectation function $\vartheta \to B'(\vartheta) = E_\vartheta X$ is strongly monotonous. For this case, we finally present just the following particular result which applies to both estimating and testing.

Proposition 1. *If conditions* (1) *and* (2) *are satisfied for* $\gamma \in \left(\frac{1}{6}, \frac{1}{4}\right]$ *then*

$$P_{\vartheta_0}^{\times n}\left(\sqrt{n}\frac{\bar{X}_n - B'(\vartheta_0)}{\sqrt{B''(\vartheta_0)}} > z_{1-\alpha(n)} + \frac{B'''(\vartheta_0)}{6\sqrt{n}(B''(\vartheta_0))^{3/2}}z_{1-\alpha(n)}^2\right) \sim \alpha(n), n \to \infty.$$

5 Sketch of proofs

Proof of Theorems 1 and 2. If condition (2) is satisfied then $x = z_{1-\alpha(n)} = o(n^\gamma), n \to \infty$ for $\gamma \in \left(\frac{1}{6}, \frac{3}{10}\right]$, and if (1) then, according to (Linnik 1961; Nagaev 1965), $P_\mu(T_n > x) \sim f_{n,s}^{(X)}(x), x \to \infty$ where $f_{n,s}^{(X)}(x) = \frac{1}{\sqrt{2\pi}x} \exp\left\{-\frac{x^2}{2} + \frac{x^3}{\sqrt{n}} \sum_{k=0}^{s-1} a_k \left(\frac{x}{\sqrt{n}}\right)^k\right\}$ and s is an integer satisfying $\frac{s}{2(s+2)} < \gamma \le \frac{s+1}{2(s+3)}$, i.e. $s = 1$ if $\gamma \in \left(\frac{1}{6}, \frac{1}{4}\right]$ and $s = 2$ if $\gamma \in \left(\frac{1}{4}, \frac{3}{10}\right]$. Here, the constants $a_0 = \frac{g_1}{6}$, $a_1 = \frac{g_2 - 3g_1^2}{24}$ are due to the skewness g_1 and kurtosis g_2 of X. Note that $\frac{g_1 x^2}{6\sqrt{n}} = o(x)$ because $x = o(n^{1/2})$, thus $x + \frac{g_1 x^2}{6\sqrt{n}} = o(n^\gamma)$, and $P_\mu\left(T_n > x + \frac{g_1 x^2}{6\sqrt{n}}\right) \sim f_{n,1}\left(x + \frac{g_1 x^2}{6\sqrt{n}}\right)$. Hence, $P_\mu\left(T_n > x + \frac{g_1 x^2}{6\sqrt{n}}\right) \sim 1 - \Phi(x)$. Similarly, $P_\mu(T_n > z_{1-\alpha(n)}(s)) \sim \alpha(n)$, $s = 1, 2$. Further, $P_{\mu_{1,n}(s)}(T_n \le z_{1-\alpha(n)}(s))$

$$= P_{\mu_{1,n}(s)}\left(\frac{\sqrt{n}}{\sigma}(\bar{X}_n - \mu_{1,n}(s)) < z_{1-\alpha(n)}(s) - \frac{\sqrt{n}}{\sigma}(\mu_{1,n}(s) - \mu)\right) = P_0\left(\frac{\sqrt{n}}{\sigma}\bar{X}_n < z_{\beta(n)}(s)\right).$$

The latter equality holds because $\{P_\mu, \mu \in (-\infty, \infty)\}$ is assumed to be a shift family. It follows that $P_{\mu_{1,n}(s)}(T_n \le z_{1-\alpha(n)}(s))$

$$= P_0\left(\frac{\sqrt{n}}{\sigma}(-\bar{X}_n) \ge z_{1-\beta(n)} + \frac{-g_1}{6\sqrt{n}}z_{1-\beta(n)}^2 + I_{\{2\}}(s)\frac{3g_2 - 4g_1^2}{72n}z_{1-\beta(n)}^3\right).$$

Note that $-g_1, g_2$ are skewness and kurtosis of $-X_1$. Thus,

$$P_{\mu_{1,n}(s)}\left(T_n \le z_{1-\alpha(n)}(s)\right) \sim f_{n,s}^{(-X)}(z_{1-\beta(n)}(s)) \sim \beta(n), n \to \infty.$$

Because $P_\mu(ACI^u$ does not cover $\mu) = P_\mu(T_n > z_{1-\alpha(n)}(s))$ and $P_{\mu_{1,n}(s)}(ACI^u$ covers $\mu) = P_{\mu_{1,n}(s)}(T_n \le z_{1-\alpha(n)}(s))$, the theorems are proved.

Proof of Remark 1. The first statement of the remark follows from

$$P_\mu\left(\mu > \bar{X}_n + \sigma z_{1-\alpha(n)}^-/\sqrt{n}\right) = P_\mu\left(\sqrt{n}(-\bar{X}_n + \mu)/\sigma > z_{1-\alpha(n)}^-\right)$$

and the second one from

$$P_{\mu_{1,n}^-(s)}\left(\mu < \bar{X}_n + \sigma z_{1-\alpha(n)}^-/\sqrt{n}\right) = P_0\left(\bar{X}_n > \mu - \mu_{1,n}^-(s) - \sigma z_{1-\alpha(n)}^-/\sqrt{n}\right)$$

$$= P_0\left(\sqrt{n}\bar{X}_n/\sigma > z_{1-\beta(n)}(s)\right).$$

Proof of Theorem 3. We start from the well known relations

$$\alpha = 1 - \Phi(z_{1-\alpha}) = \left(1 + O\left(\frac{1}{z_{1-\alpha}^2}\right)\right)\frac{1}{\sqrt{2\pi}z_{1-\alpha}}e^{-\frac{z_{1-\alpha}^2}{2}}, \alpha \to 0.$$

The solution to the approximative quantile equation $\alpha = \frac{1}{\sqrt{2\pi}x}e^{-\frac{x^2}{2}}$ will be denoted by $x = x_{1-\alpha}$. Let us put

$$xe^{\frac{x^2}{2}} = \frac{1}{\sqrt{2\pi}\alpha} =: y. \tag{3}$$

If $x \ge 1$ then it follows from (3) that $y \ge e^{\frac{x^2}{2}}$, hence $x^2 \le \ln(y^2)$. It follows again from (3) that $y^2 \le \ln(y^2)e^{x^2}$, thus $x^2 \ge \ln\left(\frac{y^2}{\ln y^2}\right)$. After one more such step,

$$\ln\left(\frac{y^2}{\ln y^2}\right) \le x^2 \le \ln\left[\frac{y^2}{\ln\left(\frac{y^2}{\ln y^2}\right)}\right].$$

The theorem now follows from

$$x^2 = \{\ln y^2 - \ln 2 - \ln \ln y\}\left\{1 + O\left(\frac{\ln \ln y}{(\ln y^2)^2}\right)\right\}, y \to \infty.$$

Let us remark that the inverse of the function $w \to we^w$ is called the Lambert W function. An asymptotic representation of the solution of (3) as $y \to \infty$ can therefore be derived from the more general representation (4.19) of W in (Corless et al. 1996) if one reads (3) as $we^w = y^2$. Our derivation of the particular result needed here, however, is much more elementary than the general one given in the paper just mentioned.

Proof of Theorem 4. Recognize that if $g_{n,s}(x) = o\left(\frac{1}{x}\right), x \to \infty$ then $f_{n,s}^{(+/-)(X)}(x + g_{n,s}(x)) \sim f_{n,s}^{(+/-)(X)}(x), x \to \infty$. Let us restrict to the case $s = 1$. According to (Linnik 1961),

$$P_{\mu_0}(T_n(1) > z_{1-\alpha(n)}) \sim P_{\mu_0}\left(\frac{3\sqrt{n}}{g_1} > T_n(1) > z_{1-\alpha(n)}\right).$$

The function $f_n^{(1)}(t) = t - \frac{g_1 t^2}{6\sqrt{n}}$ has a positive derivative, $f_n^{(1)'}(t) = 1 - \frac{g_1 t}{3\sqrt{n}} > 0$, if $g_1 t < 3\sqrt{n}$. Denoting there the inverse function of $f_n^{(1)}$ by $f_n^{(1)^{-1}}$, it follows $f_n^{(1)^{-1}}(x) = x + \frac{g_1 x^2}{6\sqrt{n}} + O\left(\frac{x^3}{n}\right)$ and $f_n^{(1)}\left(f_n^{(1)^{-1}}(x)\right) = x + o\left(\frac{1}{x}\right)$. Thus,

$$P_{\mu_o}(T_n(1) > z_{1-\alpha(n)}) \sim P_{\mu_o}(T_n > z_{1-\alpha(n)}(1)) \sim \alpha(n).$$

Moreover, $P_{\mu_{1n}(1;\alpha(n),\beta(n))}(T_n(1) \le z_{1-\alpha(n)}) = P_{\mu_{1n}(1;\alpha(n),\beta(n))}\left(T_n \le \left(f_n^{(1)}\right)^{-1}(z_{1-\alpha(n)})\right) =$

$P_{\mu_{1n}(1;\alpha(n),\beta(n))}\left(\sqrt{n}\frac{\bar{X}_n - \mu_{1n}(1)}{\sigma} \le z_{1-\alpha(n)}(1) + \frac{z_{1-\alpha(n)}^2 g_1}{6\sqrt{n}} + O\left(\frac{z_{1-\alpha(n)}^3}{n}\right) - \sqrt{n}\frac{\mu_{1n}(1) - \mu_0}{\sigma}\right) \sim$

$f_{n,1}^{(-X)}\left(-z_{\beta(n)}(1) + O\left(\frac{z_{1-\alpha(n)}^3}{n}\right)\right) \sim 1 - \Phi(z_{1-\beta(n)}) = \beta(n).$

Proof of Proposition 1. Because

$$P_{\mu_0}(reject\ H_0) = P_{\mu_0}(ACI^u(1)\ does\ not\ cover\ \mu_0)$$

it follows by Theorem 1 that

$$P_{\mu_0}(reject\ H_0) = P_{\mu_0}\left(\bar{X}_n - \frac{\sigma}{\sqrt{n}}z_{1-\alpha}(1) > \mu_0\right) = P_{\mu_0}\left(\sqrt{n}\frac{\bar{X}_n - \mu_0}{\sigma} > z_{1-\alpha}(1)\right).$$

With $P_{\mu_0} = P_{\vartheta_0}^{\times n}, \mu_0 = B'(\vartheta_0), \sigma^2 = B''(\vartheta_0)$ and $B'''(\vartheta_0)/(B''(\vartheta_0))^{3/2} = g_1$, the proof of Proposition 1 is finished.

References
Brown, LD: Fundamentals of statistical exponential families. IMS, Lecture Notes and Monograph Series. Hayward, CA (1986)
Corless, RM, Gonnet, GH, Hare, DEG, Jeffrey, DJ, Knuth, DE: On the Lambert W-Funktion. Adv. Comp. Math. **5**, 329–359 (1996)
Ibragimov, IA, Linnik, YW: Independent and stationary sequence. Walters, Nordhoff. Translation from Russian edition, 1965 (1971)

Linnik, YV: Limit theorems for sums of independent variables taking into account large deviations. I-III. Theor. Probab. Appl. **6** (1961). 131–148, 345–360; 7 (1962), 115–129

Nagaev, SV: Some limit theorems for large deviations. Theory Probab. Appl. **10**, 214–235 (1965)

Osipov, LV: Multidimensional limit theorems for large deviations. Theory Probab. Appl. **20**, 38–56 (1975)

Petrov, VV: Asymptotic behaviour of probabilities of large deviations. Theor. Probab. Appl. **13**, 408–420 (1968)

Richter, W-D: Skewness-kurtosis controlled higher order equivalent decisions. Open Stat. Probability J. **7**, 1–9 (2016). doi:10.2174/1876527001607010001