

METHODOLOGY

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A class of continuous bivariate distributions with linear sum of hazard gradient components

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Abstract

The main purpose of this article is to characterize a class of bivariate continuous non-negative distributions such that the sum of the components of underlying hazard gradient vector is a linear function of its arguments. It happens that this class is a stronger version of the Sibuya-type bivariate lack of memory property. Such a class is allowed to have only certain marginal distributions and the corresponding restrictions are given in terms of marginal densities and hazard rates. We illustrate the methodology developed by examples, obtaining two extended versions of the bivariate Gumbel's law.

Keywords: Bivariate hazard gradient, Bivariate lack of memory property, Characterization, Gumbel's bivariate exponential, Marshall-Olkin model, Sibuya's dependence function

MSC code: Primary: 62H05, 60E05

1 Introduction and motivation

Marshall and Olkin (1967) introduced the classical bivariate lack of memory property (to be denoted by $BLMP_1$) via relation

$$S_{X_1, X_2}(x_1 + t, x_2 + t) = S_{X_1, X_2}(x_1, x_2)S_{X_1, X_2}(t, t) \quad (1)$$

for all $x_1, x_2 \geq 0$ and $t > 0$, where $S_{X_1, X_2}(x_1, x_2) = P(X_1 > x_1, X_2 > x_2)$ is the joint survival function of non-negative continuous random vector (X_1, X_2) . The functional Eq. (1) tells us that, independently of t , the $BLMP_1$ preserves the distribution of both (X_1, X_2) and its residual lifetime vector $\mathbf{X}_t = [(X_1 - t, X_2 - t) | X_1 > t, X_2 > t]$.

The only bivariate distribution with exponential marginals which possesses $BLMP_1$ has a joint survival function given by

$$S_{X_1, X_2}(x_1, x_2) = \exp\{-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 \max(x_1, x_2)\}, \quad x_1, x_2 \geq 0, \quad (2)$$

where $\lambda_i \geq 0$, $i = 1, 2, 3$. The Marshall-Olkin (MO) bivariate exponential distribution (2) is generated by the stochastic representation

$$(X_1, X_2) = [\min(T_1, T_3), \min(T_2, T_3)], \quad (3)$$

where T_i are independent and exponentially distributed random variables with parameters $\lambda_i > 0$, $i = 1, 2, 3$. Unless $\lambda_3 = 0$, the distribution (2) exhibits singularity along the line $x_1 = x_2$ and its contribution to $S_{X_1, X_2}(x_1, x_2)$ is $\alpha = P(X_1 = X_2) = \frac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3}$, e.g. Marshall and Olkin (1967). Hence, the MO bivariate exponential distribution is

not absolutely continuous and does not have a probability density with respect to the two-dimensional Lebesgue measure.

Apart from MO bivariate exponential distribution, other known solutions of (1) are the bivariate distributions obtained by Freund (1961), Block and Basu (1974), Proschan and Sullo (1974), Friday and Patil (1977) and all distributions considered by Kulkarni (2006). Consult Chapter 10 in Balakrishnan and Lai (2009) for a related discussion as well.

Johnson and Kotz (1975) introduced another version of the bivariate lack of memory property (the local bivariate lack of memory property), which was rediscovered by Roy (2002) and named $BLMP_2$. The authors require conditional distributions $\{X_1 \mid X_2 > x_2\}$ and $\{X_2 \mid X_1 > x_1\}$ to possess (preserve) the univariate lack of memory property. The only absolutely continuous distribution with such a property is the Gumbel's type I bivariate exponential distribution given by

$$S_{X_1, X_2}(x_1, x_2) = \exp\{-\lambda_1 x_1 - \lambda_2 x_2 - \theta \lambda_1 \lambda_2 x_1 x_2\}, \quad x_1, x_2 \geq 0 \tag{4}$$

where $\lambda_i > 0, i = 1, 2$ and $\theta \in [0, 1]$, see Gumbel (1960). It is negative quadrant dependent since $S_{X_1, X_2}(x_1, x_2) \leq S_{X_1}(x_1)S_{X_2}(x_2)$ for all $x_1, x_2 \geq 0$.

If the first partial derivatives of $S_{X_1, X_2}(x_1, x_2)$ exist, one can define $BLMP_1$ and $BLMP_2$ alternatively. We will explore the conditional failure (hazard) rates defined by

$$r_i(x_1, x_2) = \frac{\partial}{\partial x_i} [-\ln S_{X_1, X_2}(x_1, x_2)], \quad i = 1, 2.$$

Marshall (1975) and Johnson and Kotz (1975) interpreted $r_i(x_1, x_2)$ as the conditional failure rate of X_i evaluated at x_i under condition that $X_j > x_j$ for $i = 1, 2, i \neq j$. Equivalently, the conditional hazard rates are the univariate hazard rates of conditional distributions of each variate, given certain inequality of the remainder.

Marshall (1975) named the vector $\mathbf{R}(x_1, x_2) = (r_1(x_1, x_2), r_2(x_1, x_2))$ the hazard gradient of the distribution (X_1, X_2) . The hazard gradient vector $\mathbf{R}(x_1, x_2)$ (if exists) uniquely determines the bivariate distribution by means of line integral

$$S_{X_1, X_2}(x_1, x_2) = \exp \left\{ - \int_{\mathcal{C}} \mathbf{R}(\mathbf{z}) d\mathbf{z} \right\}, \tag{5}$$

where \mathcal{C} is any sufficiently smooth continuous path beginning at $(0, 0)$ and terminating at (x_1, x_2) . Equation (5) holds provided that along the path of integration $S_{X_1, X_2}(x_1, x_2)$ is absolutely continuous and $\mathbf{R}(x_1, x_2)$ exists for almost all x_1 and x_2 , see Marshall (1975). The term "almost all" means that the set where the partial derivative does not exist has insignificant measure in the first quadrant, i.e. has a two-dimensional Lebesgue measure zero. See details and a multivariate version in Marshall (1975).

All bivariate distributions which have first partial derivatives and possess $BLMP_1$ are characterized by the equation

$$r_1(x_1, x_2) + r_2(x_1, x_2) = a_0$$

for almost all $x_1, x_2 \geq 0$, where a_0 is a non-negative constant, see Theorem 2 in Kulkarni (2006). In particular, the last relation is valid for the MO bivariate exponential distribution (2) with $a_0 = \lambda_1 + \lambda_2 + \lambda_3$ for all $x_1 \neq x_2$, since

$$\mathbf{R}(x_1, x_2) = (r_1(x_1, x_2), r_2(x_1, x_2)) = \begin{cases} (\lambda_1 + \lambda_3, \lambda_2), & \text{if } x_1 > x_2, \\ \text{does not exist,} & \text{if } x_1 = x_2, \\ (\lambda_1, \lambda_2 + \lambda_3), & \text{if } x_1 < x_2. \end{cases}$$

Thus, the sum $r_1(x_1, x_2) + r_2(x_1, x_2)$ is not defined along the line $x_1 = x_2$ having a zero two-dimensional Lebesgue measure. The reason is that the conditional probability $P(X_i > x_i | X_j > x_j)$ $i, j = 1, 2, i \neq j$, experiences a cusp at $x_1 = x_2$, i.e. it is continuous but not differentiable at $x_1 = x_2$ and the corresponding failure rate $r_i(x_1, x_2)$ is not defined, see Singpurwalla (2006), page 99.

In fact, Johnson and Kotz (1975) identify $BLMP_2$ by a local “constancy” of the failure rates $r_i(x_1, x_2)$, $i = 1, 2$, see their sections 3(iv) and 5.4. For distribution (4) one gets $r_i(x_1, x_2) = \lambda_i + a_1 x_{3-i}$, $i = 1, 2$, where a_1 is a non-negative constant. Thus, substituting $a_0 = \lambda_1 + \lambda_2$ and $a_1 = \theta \lambda_1 \lambda_2$, the sum of conditional hazard rates in $BLMP_2$ case is specified by

$$r_1(x_1, x_2) + r_2(x_1, x_2) = a_0 + a_1 x_1 + a_1 x_2$$

for all $x_1, x_2 \geq 0$.

It would be challenging to link $BLMP_1$ and $BLMP_2$ in a new class of bivariate continuous distributions, to be denoted by $\mathcal{L}(\mathbf{x}; \mathbf{a})$, satisfying relation

$$r(x_1, x_2) = r_1(x_1, x_2) + r_2(x_1, x_2) = a_0 + a_1 x_1 + a_2 x_2 \tag{6}$$

for almost all $x_1, x_2 \geq 0$, where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{a} = (a_0, a_1, a_2)$ is a parameter vector with non-negative elements, including the possibility $a_1 \neq a_2$.

To show one more member of the class $\mathcal{L}(\mathbf{x}; \mathbf{a})$, let us consider joint survival function

$$S_{X_1, X_2}(x_1, x_2) = \exp \{ -\lambda_1 x_1^2 - \lambda_2 x_2^2 - \lambda_3 \max(x_1, x_2) \}, \quad x_1, x_2 \geq 0, \tag{7}$$

which has a singular component along the line $x_1 = x_2$ and meets (3). One can verify that relation (6) is satisfied by setting $a_0 = \lambda_3 > 0$, $a_1 = 2\lambda_1 > 0$ and $a_2 = 2\lambda_2 > 0$. The distribution given by (7) belongs to the generalized Marshall-Olkin (GMO) distributions introduced by Li and Pellerey (2011). The random variables T_1, T_2 and T_3 in (3) are assumed to be independent in the class of GMO distributions, relaxing Marshall-Olkin assumption of exponential marginality. It is direct to check that $S_{X_1, X_2}(x_1, x_2) \geq S_{X_1}(x_1)S_{X_2}(x_2)$ for all $x_1, x_2 \geq 0$, i.e. the bivariate distribution in (7) is positive quadrant dependent.

Thus, one can find many examples of bivariate continuous distributions possessing $BLMP_1$ and $BLMP_2$ represented by (1) and (4), respectively, as well as those exhibiting positive or negative quadrant dependence that belong to the class $\mathcal{L}(\mathbf{x}; \mathbf{a})$ specified by (6). The class $\mathcal{L}(\mathbf{x}; \mathbf{a})$ is composed of non-negative bivariate distributions which are absolutely continuous or continuous with singularity along the line $L = \{x_1 = x_2 \geq 0\}$ such that the sum of the components of underlying hazard gradient is a linear function of both arguments x_1 and x_2 .

It happens that the class $\mathcal{L}(\mathbf{x}; \mathbf{a})$ (therefore both $BLMP_1$ and $BLMP_2$) is a particular case of the Sibuya-type bivariate lack of memory property recently introduced by Pinto and Kolev (2015c) as follows.

Definition 1. The non-negative continuous bivariate distribution (X_1, X_2) with marginal survival functions $S_{X_1}(x_1)$ and $S_{X_2}(x_2)$ possesses *Sibuya-type bivariate lack of memory property* (to be abbreviated S-BLMP), if and only if

$$\frac{S_{X_i}(x_1, x_2)}{S_{X_{1i}}(x_1)S_{X_{2i}}(x_2)} = \frac{S_{X_1, X_2}(x_1, x_2)}{S_{X_1}(x_1)S_{X_2}(x_2)} \tag{8}$$

for all $x_1, x_2, t \geq 0$, where $S_{X_t}(x_1, x_2)$ is the joint survival function of residual lifetime vector \mathbf{X}_t and $S_{X_{it}}(x_i)$ are its marginal survival functions, $i = 1, 2$.

The S-BLMP is a new concept. Definition 1 tells us that the random vector (X_1, X_2) and its residual lifetime vector \mathbf{X}_t should share, for all $t \geq 0$, the same dependence function $D_{X_1, X_2}(x_1, x_2)$ introduced by Sibuya (1960) as follows

$$D_{X_1, X_2}(x_1, x_2) = \ln \frac{S_{X_1, X_2}(x_1, x_2)}{S_{X_1}(x_1)S_{X_2}(x_2)}.$$

We will refer to $D_{X_1, X_2}(x_1, x_2)$ as ‘‘Sibuya’s dependence function’’ hereafter. It exhibits interesting connections with important cases of dependencies, see Kolev (2016) for related facts.

The article is organized as follows. In Section 2 we first discuss the S-BLMP and justify our attention to its stronger version: the linear Sibuya-type BLMP introduced by Pinto (2014), see Definition 2. We show its equivalence with the class $\mathcal{L}(\mathbf{x}; \mathbf{a})$ and present characterizations. Naturally, Theorems 1 and 2 are consequences of corresponding statements in Pinto and Kolev (2015c). It will be recognized in Section 3 that only certain marginal distributions of the class $\mathcal{L}(\mathbf{x}; \mathbf{a})$ are allowed. The corresponding conditions in terms of marginal densities and hazard rates are presented. Restrictions of parameters a_0, a_1 and a_2 are given in Theorem 3 and Proposition 1. As a result, one would be able to generate members of the class $\mathcal{L}(\mathbf{x}; \mathbf{a})$, some of them being extensions of Gumbel’s law (4), see examples 1A and 3. A discussion and conclusions finalize the paper.

2 A stronger version of S-BLMP

The marginal survival functions of residual lifetime vector \mathbf{X}_t are given by

$$S_{X_{1t}}(x_1) = \frac{S_{X_1, X_2}(x_1 + t, x_2)}{S_{X_1, X_2}(t, t)} \quad \text{and} \quad S_{X_{2t}}(x_2) = \frac{S_{X_1, X_2}(x_1, x_2 + t)}{S_{X_1, X_2}(t, t)}.$$

Therefore, the relation (8) can be rewritten as

$$S_{X_1, X_2}(x_1 + t, x_2 + t) = S_{X_1, X_2}(x_1, x_2)S_{X_1, X_2}(t, t)B(x_1, x_2; t), \tag{9}$$

for all $x_1, x_2, t \geq 0$, where the continuous function $B(x_1, x_2; t) = \frac{S_{X_{1t}}(x_1)S_{X_{2t}}(x_2)}{S_{X_1}(x_1)S_{X_2}(x_2)}$ is such that $B(x_1, x_2; 0) = B(0, 0; t) = 1$. The function $B(x_1, x_2; t)$ is named ‘‘aging factor’’, see page 457 in Balakrishnan and Lai (2009).

Without some simplifying assumptions on $B(x_1, x_2; t)$ the class of bivariate distributions possessing S-BLMP is too cumbersome to be of use. General characterizations of the S-BLMP in terms of functional equations involving Sibuya’s dependence function are presented by Pinto and Kolev (2015c), see their Lemma 1 for example.

In order to get useful models, one should investigate a stronger version of S-BLMP. In this paper we will perform a detailed analysis for a particular aging function of the form $B(x_1, x_2; t) = \exp\{-a_1x_1t - a_2x_2t\}$, where a_1 and a_2 are given nonnegative constants. One can immediately recognize that in addition to relation (8) in Definition 1 one must assume that

$$S_{X_{it}}(x_i) = S_{X_i}(x_i) \exp\{-a_ix_it\} \quad \text{for} \quad a_i \geq 0, \quad i = 1, 2. \tag{10}$$

Thus, we arrive to the stronger version of the S-BLMP, i.e. the linear Sibuya-type BLMP introduced by Pinto (2014) as follows.

Definition 2. The non-negative continuous bivariate distribution (X_1, X_2) possesses *linear Sibuya BLMP* (to be abbreviated LS-BLMP), if and only if (8) and (10) are satisfied for all $x_1, x_2, t \geq 0$ and $a_i \geq 0, i = 1, 2$.

Therefore, the bivariate continuous distributions possessing LS-BLMP can be equivalently represented by relation

$$S_{X_1, X_2}(x_1 + t, x_2 + t) = S_{X_1, X_2}(x_1, x_2)S_{X_1, X_2}(t, t) \exp\{-a_1x_1t - a_2x_2t\} \tag{11}$$

for all $x_1, x_2 \geq 0$ and $t > 0$. Really, let (11) be true. Put $x_i = 0$ in (11) to get relations (10), $i = 1, 2$. Substitute the exponent from (10) into (11) to obtain (8). Conversely, let (8) and (10) be fulfilled. Use expression (10) in the left hand side of (8) to restore (11).

To justify the choice of specific form $\exp\{-a_1x_1t - a_2x_2t\}$ of the “aging factor” in (11), let us consider a system composed by two elements with lifetimes represented by nonnegative continuous random variables X_1 and X_2 . Suppose that during the first t units of time the system is protected by breakdowns (by warranty or insurance, say). It is reasonable to assume that $t < \min(x_1, x_2)$. After those first t units of time the system can be affected by two independent “fatal shocks” governed by homogeneous Poisson processes. Assume finally that the i -th unit is damaged with intensity a_it , i.e. the corresponding shock arrival times are exponentially distributed, to be denoted $T_i \sim \text{Exp}(a_it), i = 1, 2$, see Fig. 1.

Therefore, the probability of the system survival $S_{X_1, X_2}(x_1 + t, x_2 + t)$ is given by (11). Really, the right hand side in (11) is the probability $S_{X_1, X_2}(t, t)$ that both elements survive the protected initial t units of time multiplied by $S_{X_1, X_2}(x_1, x_2) \exp\{-a_1x_1t - a_2x_2t\}$, being the probability of absence of shocks during the following x_i units of time for the i -th element, $i = 1, 2$.

The next characterization theorem gives the joint survival function of bivariate continuous distributions possessing LS-BLMP.

Theorem 1. *The continuous bivariate distribution (X_1, X_2) has LS-BLMP defined by (11) for all $x_1, x_2 \geq 0$ and $t > 0$ if and only if $S_{X_1, X_2}(x_1, x_2)$ is a non-degenerate bivariate survival function given by*

$$S_{X_1, X_2}(x_1, x_2) = \begin{cases} S_{X_1}(x_1 - x_2) \exp\{-a_0x_2 - a_1x_1x_2 - \frac{a_2 - a_1}{2}x_2^2\}, & \text{if } x_1 > x_2 \geq 0, \\ \exp\{-a_0x_1 - \frac{a_1 + a_2}{2}x_1^2\}, & \text{if } x_1 = x_2 \geq 0, \\ S_{X_2}(x_2 - x_1) \exp\{-a_0x_1 - a_2x_1x_2 - \frac{a_1 - a_2}{2}x_1^2\}, & \text{if } x_2 > x_1 \geq 0, \end{cases} \tag{12}$$

where $a_0, a_1, a_2 \geq 0$ and $S_{X_i}(x_i)$ are the marginal survival functions, $i = 1, 2$.

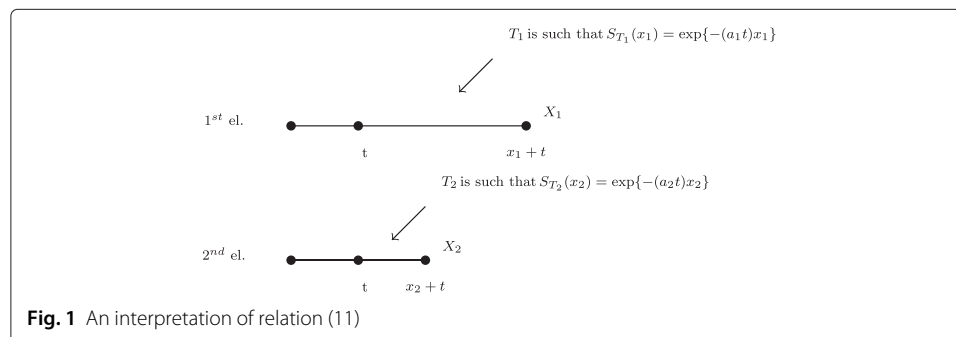


Fig. 1 An interpretation of relation (11)

Proof. Follows step by step the proof of Theorem 4 in Pinto and Kolev (2015c) with $A_i(x) = a_i x$, $i = 1, 2$. □

Theorem 1 tells us that (12) is the solution of the functional Eq. (11).

In Pinto and Kolev (2015c) is also shown that S-BLMP characterizes the distributions belonging to the class defined by relation

$$r_1(x_1, x_2) + r_2(x_1, x_2) = a_0 + A_1(x_1) + A_2(x_2), \tag{13}$$

where $a_0 > 0$ and the continuous integrable functions $A_i(x_i)$ are such that $A_i(0) = 0$ and $A_i(x_i) > -a_0$ for all $x_i > 0$, $i = 1, 2$.

One can deduce that when $B(x_1, x_2; t) = \exp\{-a_1 x_1 t - a_2 x_2 t\}$, then relation (13) transforms into (6), i.e. $A_i(x_i) = a_i x_i$, $i = 1, 2$ and we obtain the $\mathcal{L}(\mathbf{x}; \mathbf{a})$ class.

When $S_{X_1, X_2}(x_1, x_2)$ is absolutely continuous, its vector hazard gradient $\mathbf{R}(x_1, x_2)$ exists everywhere in the interior of the set

$$\mathcal{A} = \{(x_1, x_2) \in \mathbb{R}_+^2 \mid S_{X_1, X_2}(x_1, x_2) > 0\},$$

where \mathbb{R}_+^2 is the first quadrant. In other words, relation (6) is well defined for all $x_1, x_2 \geq 0$. If it happens that $S_{X_1, X_2}(x_1, x_2)$ is continuous, its vector hazard gradient $\mathbf{R}(x_1, x_2)$ is useful even when it does not exist everywhere in the interior of the set \mathcal{A} , see Marshall (1975). Because of possible singularity of the class $\mathcal{L}(\mathbf{x}; \mathbf{a})$ along the line L having zero two-dimensional Lebesgue measure, we will assume hereafter that first partial derivatives of $S_{X_1, X_2}(x_1, x_2)$ exist and are continuous in $\mathcal{A} \setminus L$.

The next characterization theorem holds for bivariate continuous distributions belonging to the class $\mathcal{L}(\mathbf{x}; \mathbf{a})$ whose survival functions possess continuous first partial derivatives, and hence continuous hazard gradient vector $\mathbf{R}(x_1, x_2)$, in $\mathcal{A} \setminus L$.

Theorem 2. *If the first partial derivatives of $S_{X_1, X_2}(x_1, x_2)$ exist and are continuous in $\mathcal{A} \setminus L$, then relation (6) is fulfilled if and only if the joint survival function can be represented by (12).*

Proof. Follows step by step the proof of Theorem 2 in Pinto and Kolev (2015c) with $A_i(x) = a_i x$, $i = 1, 2$. □

Observe that if our base relation is (11), then the LS-BLMP can be characterized by the joint survival function specified by (12), without the assumption of existence of hazard gradient vector $\mathbf{R}(x_1, x_2)$ as in Theorem 2.

Applying the Sklar’s theorem to (12), one can obtain the survival copula corresponding to the class $\mathcal{L}(\mathbf{x}; \mathbf{a})$, see Pinto and Kolev (2015a).

Remark 1. (hazard vector elements in singularity and absolutely continuous cases). The bivariate survival functions considered in Theorem 2 are not necessarily absolutely continuous (and therefore not differentiable in \mathcal{A}). In fact, we do not exclude the possibility of existence of a singular component along the line $L = \{x_1 = x_2 \geq 0\}$, i.e. it may happen that $P(X_1 = X_2) > 0$. In such a case, when (x_1, x_2) belongs to the set $\{(x_1, x_2) \in \mathbb{R}_+^2 \mid x_1 = x_2 = x\}$ the function $S_{X_1, X_2}(x_1, x_2)$ is not differentiable as well as the hazard gradient vector $\mathbf{R}(x_1, x_2)$ does not exist. But if in Theorem 2 the continuity of the

first partial derivatives of $S_{X_1, X_2}(x_1, x_2)$ holds true in \mathcal{A} , then $S_{X_1, X_2}(x_1, x_2)$ is absolutely continuous, see page 357 in Apostol (1974).

If the joint survival function $S_{X_1, X_2}(x_1, x_2)$ is *degenerate* (i.e. has degenerate marginal distributions), from (12) we get

$$S_{X_1, X_2}^{de}(x_1, x_2) = \begin{cases} \exp \left\{ -a_0x_2 - a_1x_1x_2 - \frac{a_2 - a_1}{2}x_2^2 \right\}, & \text{if } x_2 < x_1 = \text{const}, \\ \exp \left\{ -a_0x - \frac{a_1 + a_2}{2}x^2 \right\}, & \text{if } x_2 = x_1 = x = \text{const}, \\ \exp \left\{ -a_0x_1 - a_2x_1x_2 - \frac{a_1 - a_2}{2}x_1^2 \right\}, & \text{if } x_1 < x_2 = \text{const}. \end{cases}$$

Obviously, $S_{X_1, X_2}^{de}(x_1, x_2)$ does not have differentiable failure rates.

Now we will identify members of the class $\mathcal{L}(\mathbf{x}; \mathbf{a})$ with independent marginals. Hence, when $(X_1, X_2) \in \mathcal{L}(\mathbf{x}; \mathbf{a})$, we will find solutions of functional equation

$$S_{X_1, X_2}(x_1, x_2) = S_{X_1}(x_1)S_{X_2}(x_2) \quad \text{for all } x_1, x_2 \geq 0.$$

Let $r_{X_i}(x_i)$ be hazard rates of random variables X_i , $i = 1, 2$. The independence between X_1 and X_2 implies that $r_i(x_1, x_2) = r_{X_i}(x_i)$, $i = 1, 2$. Therefore, relation (6) transforms into

$$r(x_1, x_2) = r_{X_1}(x_1) + r_{X_2}(x_2) = a_0 + a_1x_1 + a_2x_2.$$

The last equation is equivalent to both

$$r_{X_1}(x_1) = \alpha_1 + a_1x_1 \quad \text{and} \quad r_{X_2}(x_2) = \alpha_2 + a_2x_2,$$

where $\alpha_1 \in [0, a_0]$ and $\alpha_2 = a_0 - \alpha_1$. Thus, we obtain the following result.

Corollary 1. *The vector (X_1, X_2) with independent marginals belongs the class $\mathcal{L}(\mathbf{x}; \mathbf{a})$ if and only if the marginal survival functions $S_{X_i}(x_i)$ have one of the following three possible analytic forms*

$$\exp\{-\alpha_i x_i\}, \quad \exp\{-0.5a_i x_i^2\} \quad \text{or} \quad \exp\{-\alpha_i x_i - 0.5a_i x_i^2\}, \quad i = 1, 2,$$

where $\alpha_1 + \alpha_2 = a_0$ with $\alpha_1 \in [0, a_0]$ and $a_i \geq 0$, $i = 1, 2$.

Therefore, the class $\mathcal{L}(\mathbf{x}; \mathbf{a})$ has only 9 members with independent marginals. It is interesting to note that if $S_X(x) = \exp\{-\alpha x_i - 0.5ax^2\}$, (see the third option in Corollary 1), then X has a linear failure rate $r_X(x) = \alpha_1 + ax$. These type of univariate distributions have been introduced by Kodlin (1967), see Sen (2006) as well. Additionally, since the two first analytic forms in Corollary 1 can be included in the third one (taking $a_i = 0$ or $\alpha_i = 0$), the members of the class $\mathcal{L}(\mathbf{x}; \mathbf{a})$ with independent marginals can also be seen as an extension of the (univariate) linear hazard rate model to the bivariate setup.

Let us note that the bivariate linear failure distribution introduced by Hangal and Ahmadi (2011) belongs to the class $\mathcal{L}(\mathbf{x}; \mathbf{a})$.

Finally, one can analyze relation (6) from a completely different point of view. Define the function

$$\psi(x_1, x_2) = r(x_1, x_2) - a_0 \quad \text{and set} \quad G_1(x_1) = \psi(x_1, 0), \quad G_2(x_2) = \psi(0, x_2).$$

The linear functions $G_i(x_i) = a_i x_i$ satisfy the functional equation

$$G_i(x_i + y_i) = G_i(x_i) + G_i(y_i) \quad \text{for all } x_i, y_i \geq 0, \quad i = 1, 2.$$

Therefore, $G_i(x)$, $i = 1, 2$, are additive functionals and only continuous solutions of Cauchy functional equation $f(x + y) = f(x) + f(y)$, see Theorem 1.1 in Sahoo and Kannappan (2011). In fact, we proved the following statement.

Lemma 1. *The class $\mathcal{L}(\mathbf{x}; \mathbf{a})$ of non-negative bivariate continuous distributions specified by relation (6) can be equivalently defined by linear (additive) functionals $G_1(x_1) = r(x_1, 0) - a_0$ and $G_2(x_2) = r(0, x_2) - a_0$, being the only continuous solutions of the Cauchy functional equation $f(x + y) = f(x) + f(y)$.*

3 Restrictions on the marginal densities or failure rates

Theorem 2 characterizes bivariate continuous distributions belonging to the class $\mathcal{L}(\mathbf{x}; \mathbf{a})$, i.e. having joint survival function $S_{X_1, X_2}(x_1, x_2)$ specified by relation (12), which imply some more restrictions on the margins of these distributions. In other words, the corresponding joint survival function is valid only for certain marginal distributions of X_1 and X_2 .

Here we will obtain the associated constraints in terms of marginal densities and hazard rates. As a result, we will get the admissible values of the parameter vector $\mathbf{a} = (a_0, a_1, a_2)$. The methodology will be illustrated by several typical examples.

3.1 Marginal density restrictions

The next statement shows the corresponding parameter constraints in terms of marginal densities when $a_1 + a_2 > 0$. The case $a_1 = a_2 = 0$, e.g. the $BLMP_1$, is detailed studied by Kulkarni (2006).

Theorem 3. *Let X_i be a random variable with absolutely continuous density $f_{X_i}(x_i)$, $i = 1, 2$. Let $a_0, a_1, a_2 \geq 0$ and $a_1 + a_2 > 0$. Then $S_{X_1, X_2}(x_1, x_2)$ in (12) is a proper bivariate survival function if and only if*

$$A(x_i, x_j) - a_i x_j + \frac{d}{dx_i} \log f_{X_i}(x_i - x_j) + a_i [x_j A(x_i, x_j) - 1] \frac{S_{X_i}(x_i - x_j)}{f_{X_i}(x_i - x_j)} \geq 0 \tag{14}$$

where $A(x_i, x_j) = a_0 + a_i x_i + (a_j - a_i) x_j$ for all $x_i \geq x_j \geq 0$, $i \neq j$, $i, j = 1, 2$. The singularity contribution into the joint survival function is given by

$$\alpha = P(X_1 = X_2) = [f_{X_1}(0) + f_{X_2}(0) - a_0] \sqrt{\frac{\pi}{2(a_1 + a_2)}} \times \exp \left\{ \frac{a_0^2}{2(a_1 + a_2)} \right\} \left[1 - \text{Erf} \left(\frac{a_0}{\sqrt{2(a_1 + a_2)}} \right) \right], \tag{15}$$

where $\text{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp\{-t^2\} dt$. In addition, the survival function specified by (12) is absolutely continuous if and only if $f_{X_1}(0) + f_{X_2}(0) = a_0$.

Proof. Let $S_{X_1, X_2}(x_1, x_2)$ be given by (12). Then, it is a proper if

$$\frac{\partial^2}{\partial x_1 \partial x_2} S_{X_1, X_2}(x_1, x_2) \geq 0,$$

After some algebra the last condition transforms into inequality (14).

Since $S_{X_1, X_2}(x_1, x_2)$ may have a singular component along the line $x_1 = x_2$, then $\alpha = P(X_1 = X_2) \in [0, 1]$. The bivariate survival function $S_{X_1, X_2}(x_1, x_2)$ in (12) will be

proper if and only if both the absolutely continuous part $S_{X_1, X_2}^{ac}(x_1, x_2)$ and the singular part $S_{X_1, X_2}^{si}(x_1, x_2)$ are survival functions and

$$S_{X_1, X_2}(x_1, x_2) = (1 - \alpha)S_{X_1, X_2}^{ac}(x_1, x_2) + \alpha S_{X_1, X_2}^{si}(\max\{x_1, x_2\})$$

for $\alpha \in [0, 1]$. An equivalent expression in terms of joint densities is given by

$$f_{X_1, X_2}(x_1, x_2) = (1 - \alpha)f_{X_1, X_2}^{ac}(x_1, x_2) + \alpha f_{X_1, X_2}^{si}(\max\{x_1, x_2\}),$$

where

$$(1 - \alpha)f_{X_1, X_2}^{ac}(x_1, x_2) = \frac{\partial^2}{\partial x_1 \partial x_2} S_{X_1, X_2}(x_1, x_2).$$

To ensure existence of $S_{X_1, X_2}(x_1, x_2)$ we should evaluate the probability α which is equivalent to impose that $1 - \alpha = P(X_1 > X_2) + P(X_2 > X_1) \in [0, 1]$, so one has to calculate the probabilities in the last sum. We have

$$P(X_1 > X_2) = \int_0^\infty \int_0^u (1 - \alpha)f_{X_1, X_2}^{ac}(u, v) \, dv \, du.$$

Computing the inner integral $I(u) = \int_0^u (1 - \alpha)f_{X_1, X_2}^{ac}(u, v) \, dv$ we get

$$I(u) = -f_{X_1}(0) \exp\left\{-a_0 u - \frac{a_2 + a_1}{2} u^2\right\} - a_1 u \exp\left\{-a_0 u - \frac{a_2 + a_1}{2} u^2\right\} + f_{X_1}(u).$$

Therefore,

$$\begin{aligned} P(X_1 > X_2) &= \int_0^\infty I(u) \, du \\ &= -f_{X_1}(0) \int_0^\infty \exp\left\{-a_0 u - \frac{a_2 + a_1}{2} u^2\right\} \, du - a_1 \int_0^\infty u \exp\left\{-a_0 u - \frac{a_2 + a_1}{2} u^2\right\} \, du + 1. \end{aligned} \tag{16}$$

In order to solve integrals in (16), we use the following two expressions taken from Gradshteyn and Ryzhik (2007) for positive constants c and d :

$$\int_0^\infty \exp\{-cu^2 - du\} \, du = \frac{1}{2} \sqrt{\frac{\pi}{c}} \exp\left\{\frac{d^2}{4c}\right\} \left[1 - \operatorname{Erf}\left(\frac{d}{2\sqrt{c}}\right)\right],$$

(see equation 3.322.2 on page 336), and

$$\int_0^\infty u \exp\{-cu^2 - du\} \, du = \frac{1}{2c} - \frac{d}{4c} \sqrt{\frac{\pi}{c}} \exp\left\{\frac{d^2}{4c}\right\} \left[1 - \operatorname{Erf}\left(\frac{d}{2\sqrt{c}}\right)\right],$$

(see equation 3.462.5 on page 365).

Substituting $c = \frac{a_1 + a_2}{2}$ and $d = a_0$ we obtain from (16)

$$\begin{aligned} P(X_1 > X_2) &= \frac{a_2}{a_1 + a_2} - \left[f_{X_1}(0) - \frac{a_0 a_1}{a_1 + a_2}\right] \sqrt{\frac{\pi}{2(a_1 + a_2)}} \exp\left\{\frac{a_0^2}{2(a_1 + a_2)}\right\} \\ &\quad \times \left[1 - \operatorname{Erf}\left(\frac{a_0}{\sqrt{2(a_1 + a_2)}}\right)\right]. \end{aligned}$$

In a similar way we get

$$\begin{aligned} P(X_2 > X_1) &= \frac{a_1}{a_1 + a_2} - \left[f_{X_2}(0) - \frac{a_0 a_2}{a_1 + a_2}\right] \sqrt{\frac{\pi}{2(a_1 + a_2)}} \exp\left\{\frac{a_0^2}{2(a_1 + a_2)}\right\} \\ &\quad \times \left[1 - \operatorname{Erf}\left(\frac{a_0}{\sqrt{2(a_1 + a_2)}}\right)\right]. \end{aligned}$$

From the last expressions and (16) we arrive to (15). To conclude the proof, observe that $S_{X_1, X_2}(x_1, x_2)$ in (12) is absolutely continuous if and only if $\alpha = 0$, which is equivalent to required condition $f_{X_1}(0) + f_{X_2}(0) = a_0$. \square

Notice that if inequality (14) is fulfilled for some $a_0 = u > 0$, then it is also satisfied for all $a_0 \geq u$. Denote by

$\tau =$ the greatest lower bound of the set of possible values of a_0 satisfying(14).

Depending on the marginal densities, it may happen in some special cases that $\tau > f_{X_1}(0) + f_{X_2}(0)$, which contradicts (15) since always $\alpha = P(X_1 = X_2) \geq 0$, i.e. $a_0 \leq f_{X_1}(0) + f_{X_2}(0)$. Such high τ -values would be outside of the parameter space \mathcal{A} of the class $\mathcal{L}(\mathbf{x}; \mathbf{a})$.

The range of possible values of a_0 are shown in Proposition 1. It is crucial for the construction of proper bivariate survival functions belonging to the class $\mathcal{L}(\mathbf{x}; \mathbf{a})$ from pre-specified marginal densities as we will see later.

Proposition 1. *Suppose $\tau \leq f_{X_1}(0) + f_{X_2}(0)$. If*

$$a_0 \in [\max\{\tau, \max\{f_{X_1}(0), f_{X_2}(0)\}, f_{X_1}(0) + f_{X_2}(0)\}, f_{X_1}(0) + f_{X_2}(0)], \tag{17}$$

then Theorem 3 is fulfilled.

Proof. In the absence of singularity (whenever $\alpha = P(X_1 = X_2) = 0$), one concludes from (15) that $f_{X_1}(0) + f_{X_2}(0) - a_0 = 0$. Therefore, always $a_0 \leq f_{X_1}(0) + f_{X_2}(0)$, which is the upper bound for a_0 in (17).

The increase of the singular contribution into $S_{X_1, X_2}(x_1, x_2)$ implies increasing of the probability $\alpha = P(X_1 = X_2)$ up to 1. Let us denote by

$$E(a_0, a_1, a_2) = a_0 \sqrt{\frac{\pi}{2(a_1 + a_2)}} \exp \left\{ \frac{a_0^2}{2(a_1 + a_2)} \right\} \left[1 - \text{Erf} \left(\frac{a_0}{\sqrt{2(a_1 + a_2)}} \right) \right].$$

It is direct to check that $0 \leq E(a_0, a_1, a_2) \leq 1$. We may represent (16) as

$$P(X_1 > X_2) = 1 - \frac{f_{X_1}(0)}{a_0} E(a_0, a_1, a_2) - \frac{a_1}{a_1 + a_2} [1 - E(a_0, a_1, a_2)].$$

The right hand side of the last equation is non-negative if $f_{X_1}(0) \leq a_0$. By analogy, from the expression for $P(X_2 > X_1)$ we obtain $f_{X_2}(0) \leq a_0$.

Notice that if $a_0 \in [\max\{f_{X_1}(0), f_{X_2}(0)\}, f_{X_1}(0) + f_{X_2}(0)]$ then $\alpha \in [0, 1]$. Finally, the lower bound in (17) can be obtained by taking into account the restriction on a_0 imposed by inequality (14) and possible related τ -values. \square

Remark 2. (absolutely continuous rule). Observe that whenever the upper bound a_U for a_0 given by (17) is attainable, i.e. if $a_U = a_0 = f_{X_1}(0) + f_{X_2}(0)$, one obtains an absolutely continuous bivariate distribution for which (6) is valid. Therefore, Eq. (15), besides representing the constraint $a_1 + a_2 > 0$, also offers a way to identify the presence of singularity.

For absolutely continuous distributions belonging to the class $\mathcal{L}(\mathbf{x}; \mathbf{a})$, it may happen that the lower and upper bound in (17) coincide, being even zero when $f_{X_1}(0) = f_{X_2}(0) =$

0, or equivalently, when $r_{X_1}(0) = r_{X_2}(0) = 0$, indicating that $a_0 = 0$. For example, consider a joint distribution given by $S_{X_1, X_2}(x_1, x_2) = \exp\{-0.5a_1x_1^2 - 0.5a_2x_2^2\}$. Observe that both $f_{X_1}(0) = f_{X_2}(0) = 0$ implying $a_0 = 0$ and $\alpha = P(X_1 = X_2) = 0$. This special absolutely continuous bivariate distribution with independent marginals case may be treated as an “exception”, compare with Corollary 1.

Finally, note that all members of the class $\mathcal{L}(\mathbf{x}; \mathbf{a})$ with independent marginals are absolutely continuous.

Remark 3. (singularity of the BLMP₁ models). In the particular case $a_1 = a_2 = 0$, the condition (14) transforms into inequality (ii) of Theorem 5.1 in Marshall and Olkin (1967). In addition, the possible interval values of a_0 in (17) are compatible with those given by Kulkarni (2006) in her Remark 1.

Remark 4. (singularity of the GMO models). Let us consider the joint survival function of (Y_1, Y_2) belonging to the class of GMO distributions and represented by (7), with $a_i = 2\lambda_i$, $i = 1, 2$, and $\lambda_3 = a_0$. It is direct to check that

$$P(Y_1 = Y_2) = a_0 \sqrt{\frac{\pi}{2(a_1 + a_2)}} \exp\left\{\frac{a_0^2}{2(a_1 + a_2)}\right\} \left[1 - \operatorname{Erf}\left(\frac{a_0}{\sqrt{2(a_1 + a_2)}}\right)\right].$$

Observe that the right hand side in the last equation is just the function $E(a_0, a_1, a_2)$ used in the proof of Proposition 1.

We noted in the proof of Theorem 3 that if the survival function $S_{X_1, X_2}(x_1, x_2)$ given by (12) is proper then $\frac{\partial^2}{\partial x_1 \partial x_2} S_{X_1, X_2}(x_1, x_2)$ should be non-negative. This condition is equivalent to the requirement

$$S_{X_1, X_2}(x_1, y_1) + S_{X_1, X_2}(x_2, y_2) - S_{X_1, X_2}(x_1, y_2) - S_{X_1, X_2}(x_2, y_1) \geq 0$$

for any two points (x_1, y_1) and (x_2, y_2) in \mathbb{R}_+^2 such that $x_1 \leq x_2$ and $y_1 \leq y_2$. For example, if $x_1 \leq x_2 \leq y_1 \leq y_2$ and $a_2 = 0$ we conclude from (12) that the last inequality regarding the joint survival function is equivalent to

$$\frac{S_{X_2}(y_1 - x_1) - S_{X_2}(y_2 - x_1)}{S_{X_2}(y_1 - x_2) - S_{X_2}(y_2 - x_2)} \leq \exp\left\{-(x_2 - x_1) \left[a_0 + \frac{a_1}{2}(x_2 + x_1)\right]\right\}.$$

In general, such constraints between marginal survival functions are not easily verified. Relations (14) and (15) in Theorem 3 give alternative conditions in terms of absolutely continuous marginal densities $f_{X_i}(x)$, $i = 1, 2$. However, depending on the complexity of the analytical form of the densities involved, it may be difficult to check these restrictions.

3.2 Marginal failure rates restrictions

The next result offers another set of equivalent constraints for parameters of $\mathcal{L}(\mathbf{x}; \mathbf{a})$, but in terms of marginal failure rates $r_{X_i}(x)$, $i = 1, 2$.

Theorem 4. *Let the marginal failure rates $r_{X_i}(x)$, $i = 1, 2$, be differentiable functions and for some nonnegative constants a_0, a_1 and a_2 , with $a_1 + a_2 > 0$, the following relations hold*

$$0 \leq r_{X_i}(x_i) \leq a_0 + a_i x_i; \tag{18a}$$

$$r_{X_i}(x_i - x_j)[A(x_i, x_j) - a_i x_j - r_{X_i}(x_i - x_j)] + \frac{d}{dx_i} r_{X_i}(x_i - x_j) + a_i[x_j A(x_i, x_j) - 1] \geq 0, \tag{18b}$$

with $A(x_i, x_j) = a_0 + a_i x_i + (a_j - a_i)x_j$, for $x_i \geq x_j \geq 0, i, j = 1, 2, i \neq j$;

$$a_0 \in [\max\{\tau, \max(r_{X_1}(0), r_{X_2}(0))\}, r_{X_1}(0) + r_{X_2}(0)], \tag{18c}$$

where τ denotes the greatest lower bound of the set of values of a_0 for which the inequality (18b) is satisfied.

Then the joint survival function $S_{X_1, X_2}(x_1, x_2)$ given by (12) is proper with marginals $S_{X_i}(x_i) = \exp(-\int_0^{x_i} r_{X_i}(u)du)$, $x_i \geq 0, i = 1, 2$. The joint distribution is absolutely continuous if and only if $a_0 = r_{X_1}(0) + r_{X_2}(0)$, otherwise it possesses a singular component.

Proof. Let $S_{X_1, X_2}(x_1, x_2)$ be given by (12) and let the univariate failure rates $r_{X_i}(x_i)$ be differentiable functions, $i = 1, 2$. Suppose that $x_2 \geq x_1 \geq 0$. Then substituting $x_1 = 0$ in

$$r_1(x_1, x_2) = \begin{cases} r_{X_1}(x_1 - x_2) + a_1 x_2, & \text{if } x_1 > x_2 \geq 0, \\ a_0 + (a_1 - a_2)x_1 + a_2 x_2 - r_{X_2}(x_2 - x_1), & \text{if } x_2 > x_1 \geq 0 \end{cases}$$

one gets $r_1(0, x_2) = a_0 + a_2 x_2 - r_{X_2}(x_2) \geq 0$ and therefore $r_{X_2}(x_2) \leq a_0 + a_2 x_2$. By analogy, we conclude that $r_{X_1}(x_1) \leq a_0 + a_1 x_1$ if $x_1 \geq x_2 \geq 0$ and (18a) is established.

All other relations are consequence of Theorem 3. □

Conditions (18a), (18b) and (18c) in Theorem 4 imply several simple practical steps that help to fix the permissible parameter space of coefficients a_0, a_1 and a_2 in $\mathcal{L}(\mathbf{x}; \mathbf{a})$. We discuss them in the next remark.

Remark 5. (parameter space of the class $\mathcal{L}(\mathbf{x}; \mathbf{a})$). Inequality (18a) says that the bivariate distributions from the class $\mathcal{L}(\mathbf{x}; \mathbf{a})$ satisfying (6) cannot have marginal distributions with failure rates $r_{X_i}(x_i)$ above the line $a_0 + a_i x_i$ for $i = 1, 2$. For example, distributions with univariate failure rate of the form $a x_i^2$, for $a > 0$ are unable to meet (6).

In addition, since $r_{X_i}(\cdot)$ is a failure rate, then $\int_0^\infty r_{X_i}(u)du = \infty$ and because of (18a) the support of $X_i, i = 1, 2$, cannot be bounded from above, i.e. has to be the entire half line $[0, \infty)$.

To summarize, the parameter space for the coefficients a_0, a_1 and a_2 satisfying (6) in terms of marginal failure rates is given by

$$\{a_0 \in [\max\{\tau, \max(r_{X_1}(0), r_{X_2}(0))\}, r_{X_1}(0) + r_{X_2}(0)], a_1, a_2 \geq 0, a_1 + a_2 > 0\},$$

because $r_{X_i}(0) = f_{X_i}(0)$ for $i = 1, 2$. Finally, note that admissible values for coefficients a_1 and a_2 may be further limited as a consequence of inequality (18b) from Theorem 4, see Example 2.

The converse of Theorem 4 also holds for non-degenerate distributions and the statement is given below.

Proposition 2. *If $S_{X_1, X_2}(x_1, x_2)$ is non-degenerate bivariate survival function given by Eq. (12) and has differentiable marginal failure rates then it must satisfy conditions (18a) to (18c) in Theorem 4.*

Proof. Follows step by step the proof of Proposition 1 in Kulkarni (2006). □

The rules established in Theorem 4 may serve as a useful guide for constructing bivariate distributions possessing property (6). The building scheme may be relaxed under additional available information regarding monotone behavior of marginal failure rates. In fact, the class of bivariate distributions $\mathcal{L}(\mathbf{x}; \mathbf{a})$ may have arbitrary combination of marginal failure rates: increasing, decreasing, constant, bathtub, etc., implying corresponding restrictions for the parameter space, of course.

3.3 Examples

The next two examples illustrate how relations in Theorem 4 can be applied to construct bivariate distributions from $\mathcal{L}(\mathbf{x}; \mathbf{a})$ with given marginal failure rates.

Example 1. (constant failure rate marginals). Assume that

$$S_{X_i}(x) = \exp\{-\lambda_i x\}, \quad \text{i.e. } X_i \sim \text{Exp}(\lambda_i) \quad \lambda_i > 0, \quad i = 1, 2.$$

Then $f_{X_i}(x) = \lambda_i \exp\{-\lambda_i x\}$, $r_{X_i}(x) = \lambda_i$ and $f_{X_i}(0) = r_{X_i}(0) = \lambda_i$, $i = 1, 2$. From (18c) we obtain the first restriction: $\max(\lambda_1, \lambda_2) \leq a_0 \leq \lambda_1 + \lambda_2$.

Let $x_1 \geq x_2$. Since $a_1 \geq 0$, inequality (18b) transforms into

$$0 \leq a_1 \leq (\lambda_1 + a_1 x_2)[a_0 - \lambda_1 + a_1(x_1 - x_2) + a_2 x_2],$$

for all $x_1 \geq x_2 \geq 0$. The function $(\lambda_1 + a_1 x_2)[a_0 - \lambda_1 + a_1(x_1 - x_2) + a_2 x_2]$ is non-decreasing and its minimum is equal to $\lambda_1(a_0 - \lambda_1)$ when $x_1 = x_2 = 0$. Therefore, $0 \leq a_1 \leq \lambda_1(a_0 - \lambda_1)$.

To find the greatest lower bound τ for which condition (18c) in Theorem 4 is true, is equivalent to verify when $\lambda_2(a_0 - \lambda_2) + \lambda_2 a_0 a_1 x_2 + \lambda_2 a_0 a_1 x_2^2 \geq 0$. The last inequality is satisfied when $a_0 \geq \lambda_2$. But we got this lower bound for a_0 already.

Analogously, when $x_2 \geq x_1$ we obtain $0 \leq a_2 \leq \lambda_2(a_0 - \lambda_2)$.

Summarizing, the parameter space is

$$\max(\lambda_1, \lambda_2) \leq a_0 \leq \lambda_1 + \lambda_2, \quad a_1 + a_2 > 0 \quad \text{and} \quad 0 \leq a_i \leq \lambda_i(a_0 - \lambda_i), \quad i = 1, 2.$$

We will consider two possible cases:

1A. The bivariate survival function will be absolutely continuous if $a_0 = \lambda_1 + \lambda_2$. Hence, $a_i = \theta_i \lambda_1 \lambda_2$ for $\theta_i \in (0, 1]$, $i = 1, 2$. With these specific parameters we get from (12) the representation

$$S_{X_1, X_2}(x_1, x_2) = \begin{cases} \exp \left\{ - \left[\lambda_1 x_1 + \lambda_2 x_2 + \lambda_1 \lambda_2 x_2 \left(\theta_1 x_1 + \frac{\theta_2 - \theta_1}{2} x_2 \right) \right] \right\}, & \text{if } x_1 \geq x_2, \\ \exp \left\{ - \left[\lambda_1 x_1 + \lambda_2 x_2 + \lambda_1 \lambda_2 x_1 \left(\theta_2 x_2 + \frac{\theta_1 - \theta_2}{2} x_1 \right) \right] \right\}, & \text{if } x_2 \geq x_1, \end{cases} \tag{19}$$

which may be named *Generalized Gumbel's bivariate exponential distribution*.

Observe, that fixing $\theta_1 = \theta_2 = \theta$ in the last relation we obtain as a particular case the Gumbel's type I bivariate exponential distribution (4).

1B. A bivariate survival function with absolutely continuous and singular components can also be constructed when $a_0 < \lambda_1 + \lambda_2$. Suppose $\lambda_1 > \lambda_2$ and let $a_0 = \lambda_1$. Notice that with this parameter choice the restrictions in (18c) are fulfilled. Hence we obtain $a_1 = 0$ and $a_2 = \theta\lambda_2(\lambda_1 - \lambda_2)$, where $\theta \in (0, 1]$. Substituting these parameter values in (12) we get

$$S_{X_1, X_2}(x_1, x_2) = \begin{cases} \exp \left\{ - \left[\lambda_1 x_1 + \frac{\theta \lambda_2 (\lambda_1 - \lambda_2)}{2} x_2^2 \right] \right\}, & \text{if } x_1 \geq x_2, \\ \exp \left\{ - [(\lambda_1 - \lambda_2)x_1 + \lambda_2 x_2] \right. \\ \left. \times \exp \left\{ - \left[\theta \lambda_2 (\lambda_1 - \lambda_2) x_1 \left(x_2 - \frac{x_1}{2} \right) \right] \right\} \right\}, & \text{if } x_2 \geq x_1. \end{cases}$$

In what follows, we will build a bivariate distribution with increasing marginal failure rates.

Example 2. (increasing failure rate marginals). Consider

$$S_{X_i}(x) = \exp \left\{ -\lambda_i x^2 - \lambda_3 x \right\} \quad \text{for } x \geq 0, \lambda_i > 0, \lambda_3 > 0, i = 1, 2.$$

Since $f_{X_i}(x) = (2\lambda_i x + \lambda_3) \exp \left\{ -\lambda_i x^2 - \lambda_3 x \right\}$, then $r_{X_i}(x) = 2\lambda_i x + \lambda_3, i = 1, 2$, and the marginals have increasing failure rate. First limitations on the parameter space come from inequalities (18a) and (18c), i.e. $a_0 \geq \lambda_3$ and $a_i \geq 2\lambda_i, i = 1, 2$.

When $x_1 \geq x_2 \geq 0$, from (18b) we get a nonnegative increasing in x_1 and x_2 function

$$2\lambda_1 - a_1 - [2\lambda_1(x_1 - x_2) + \lambda_3 + a_1 x_2] [(2\lambda_1 - a_1)(x_1 - x_2) + \lambda_3 - a_0 - a_2 x_2] \geq 0$$

with a minimum at the point (0, 0). Hence we obtain $a_1 \leq 2\lambda_1 + \lambda_3(a_0 - \lambda_3)$.

Analogously, for $x_1 \geq x_2 \geq 0$, we get $2\lambda_2 \leq a_2 \leq 2\lambda_2 + \lambda_3(a_0 - \lambda_3)$.

Summarizing, we have the constraints

$$\lambda_3 \leq a_0 \leq 2\lambda_3 \text{ and } 2\lambda_i \leq a_i \leq 2\lambda_i + \lambda_3(a_0 - \lambda_3), i = 1, 2.$$

2A. If $a_0 = 2\lambda_3$ we obtain an absolutely continuous bivariate survival function. In this case $a_i = 2\lambda_i + \theta_i \lambda_3^2, \theta_i \in [0, 1], i = 1, 2$ and letting these values in (12) one gets

$$S_{X_1, X_2}(x_1, x_2) = \begin{cases} \exp \left\{ - [\lambda_1 x_1^2 + \lambda_3 x_1 + \lambda_2 x_2^2] \right\} \\ \times \exp \left\{ - \left[\lambda_3 x_2 + \lambda_3^2 x_2 \left(\theta_1 x_1 + \frac{\theta_2 - \theta_1}{2} x_2 \right) \right] \right\}, & \text{if } x_1 \geq x_2, \\ \exp \left\{ - [\lambda_2 x_2^2 + \lambda_3 x_2 + \lambda_1 x_1^2] \right\} \\ \times \exp \left\{ - \left[\lambda_3 x_1 + \lambda_3^2 x_1 \left(\theta_2 x_2 + \frac{\theta_1 - \theta_2}{2} x_1 \right) \right] \right\}, & \text{if } x_2 \geq x_1. \end{cases}$$

Observe that the expression of the joint survival function involves a complete second degree polynomial in the exponent. In addition, notice that $\theta_1 = \theta_2 = 0$ implies independence between X_1 and X_2 .

2B. A bivariate survival function having absolutely continuous and singular component can also be captured substituting

$$a_0 = (1 + \theta_0)\lambda_3, \quad \theta_0 \in [0, 1), \quad a_i = 2\lambda_i + \theta_0 \theta_i \lambda_3^2 \quad \text{and} \quad \theta_i \in [0, 1], i = 1, 2$$

in (12). Let $\theta_0 = 0$ in the corresponding expression to get relation (7), i.e the distribution from the class of GMO distributions, see Li and Pellerey (2011).

Example 3. (“min” operation based construction). Let the survival function of the bivariate random vector (Y_1, Y_2) follow Gumbel’s type I bivariate exponential distribution given by (4) which is a member of the class $\mathcal{L}(\mathbf{x}; \mathbf{a})$. Assume that $Y_3 \sim \text{Exp}(\lambda_3)$ is independent of (Y_1, Y_2) . Therefore, $(X_1, X_2) = [\min(Y_1, Y_3), \min(Y_2, Y_3)]$ belongs to $\mathcal{L}(\mathbf{x}; \mathbf{a})$ as well and its survival function is given by

$$S_{X_1, X_2}(x_1, x_2) = \begin{cases} \exp\{-(\lambda_1 + \lambda_3)x_1 - \lambda_2 x_2 - \theta \lambda_1 \lambda_2 x_1 x_2\}, & \text{if } x_1 \geq x_2, \\ \exp\{-\lambda_1 x_1 - (\lambda_2 + \lambda_3)x_2 - \theta \lambda_1 \lambda_2 x_1 x_2\}, & \text{if } x_2 \geq x_1. \end{cases} \tag{20}$$

Notice that $S_{Y_1, Y_2}(x_1, x_2)$ is absolutely continuous, but $S_{X_1, X_2}(x_1, x_2)$ displays a singular component along the line $x_1 = x_2$. The expression (20) is given by Pinto and Kolev (2015c) in their Example 1.

It is worth noting that $S_{X_1, X_2}(x_1, x_2)$ given by (20), despite being continuous (but not absolutely continuous), preserves the local constancy of the failure rates $r_1(x_1, x_2)$ and $r_2(x_1, x_2)$ in a very similar fashion as Gumbel’s bivariate distribution (4) does, i.e. $r_i(x_1, x_2) = \lambda_i + a_1 x_{3-i}$, $i = 1, 2$. The hazard components of (20) are given by

$$r_i(x_1, x_2) = \begin{cases} \lambda_i + \lambda_3 + \theta \lambda_1 \lambda_2 x_{3-i}, & \text{if } x_i > x_{3-i}, \\ \text{does not exist} & \text{if } x_1 = x_2, \\ \lambda_i + \theta \lambda_1 \lambda_2 x_{3-i}, & \text{if } x_i < x_{3-i} \end{cases}$$

for $i = 1, 2$. Therefore, we may consider (20) as a *Gumbel’s extended bivariate exponential distribution with a singularity* along the line $x_1 = x_2$. Substituting $\lambda_3 = 0$ in (20), one will get absolutely continuous Gumbel’s version (4).

Remark 6. (expanding Gumbel’s bivariate law). The construction in Example 3 incorporates a singular component into the resulting distribution, which belongs to a wider class of Extended Marshall Olkin (EMO) bivariate distributions introduced by Pinto and Kolev (2015b). In fact, we assume that T_1 and T_2 are dependent random variables, but independent of T_3 in stochastic representation (3). Particular members of the EMO-class are the MO bivariate exponential distribution satisfying (3) and the GMO distributions (remind that the joint distribution given by (7) is member of the GMO-class).

4 Discussion and conclusions

In this paper we study a stronger version of S-BLMP introduced by in Pinto and Kolev (2015c), see Definition 1. We define the class $\mathcal{L}(\mathbf{x}; \mathbf{a})$ and characterize it by the following equivalent relations $\mathcal{L}(\mathbf{x}; \mathbf{a}) \Leftrightarrow (6) \Leftrightarrow (12)$. In addition,

$$\text{Lemma 1} \Leftrightarrow \mathcal{L}(\mathbf{x}; \mathbf{a}) \Leftrightarrow \text{LS} - \text{BLMP} \Leftrightarrow (11).$$

Thus, the class $\mathcal{L}(\mathbf{x}; \mathbf{a})$ might be treated as a key tool to deepen the BLMP-notion giving possibility to model the aging phenomena in the complement to the “non-aging” one, which fixes the world on Eqs. (1) or (4), via $BLMP_1$ and $BLMP_2$ correspondingly. Our new base equations are (6) or (11). We are convinced that the class introduced is promising in modeling dynamic aging dependence, being much more realistic than the virtual “non-aging world”. The class $\mathcal{L}(\mathbf{x}; \mathbf{a})$ includes symmetric and asymmetric continuous distributions with possible singularity; those which are positive or negative quadrant

dependent; distributions from the GMO and EMO classes, etc. This huge variety of bivariate distributions would help to choose the “right” model consistent with the physical nature of the observations. The selection of bivariate distribution to be used should depend on considerations involving both the physical scenario at hand, and the properties of chosen distribution.

The seminal Gumbel’s type I bivariate exponential distribution given by (4) has a central place in probability theory, being object of many characterization results. To count several of them, it is: the only absolutely continuous bivariate distribution possessing $BLMP_2$; belongs to the class $\mathcal{L}(\mathbf{x}; \mathbf{a})$; a key bivariate extreme value distribution, etc. We got two extensions of the Gumbel’s type I distribution: one being absolutely continuous and the other having a singular component, represented by (19) and (20) respectively, consult Remark 6 as well. We do believe that further characterizations based on those relations will elevate the generalized Gumbel’s laws as a starting point and a base for obtaining new bivariate models, with higher flexibility and chances to better model the genuine dependence structure.

We did not consider in this article inference procedures related to the model (6), nor its application for real data set. But, in Pinto and Kolev (2015b) we perform a Bayesian analysis using the EMO distribution (20) (being a member of the class $\mathcal{L}(\mathbf{x}; \mathbf{a})$) for a soccer data with ties studied by Meintanis (2007) and many other authors. According to the Deviance Information Criterion, our model (20) presented better fit than the Marshall-Olkin bivariate Weibull distribution, recently introduced by Kundu and Gupta (2013), who analyzed the same data set.

In general, one may use techniques for estimation of bivariate density function with partially differentiable kernels, e.g. Scott (1992). Another option is to apply the Kaplan-Meier estimate of bivariate survival function, even in the case of censoring following Dabrowska (1988), for example. Once the model is selected, goodness of the fit can be tested with conventional methods.

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