

REVIEW

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Recent developments on the moment problem

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Abstract

We consider univariate distributions with finite moments of all positive orders. The moment problem is to determine whether or not a given distribution is uniquely determined by the sequence of its moments. There is a huge literature on this classical topic. In this survey, we will focus only on the recent developments on the *checkable* moment-(in)determinacy criteria including Cramér's condition, Carleman's condition, Hardy's condition, Krein's condition and the growth rate of moments, which help us solve the problem more easily. Both Hamburger and Stieltjes cases are investigated. The former is concerned with distributions on the whole real line, while the latter deals only with distributions on the right half-line. Some new results or new simple (direct) proofs of previous criteria are provided. Finally, we review the most recent moment problem for products of independent random variables with different distributions, which occur naturally in stochastic modelling of complex random phenomena.

Keywords: Hamburger moment problem, Stieltjes moment problem, Cramér's condition, Carleman's condition, Krein's condition, Hardy's condition

AMS Subject Classification: 60E05; 44A60

1 Introduction

The moment problem is a classical topic over one century old (Stieltjes 1894, 1895, Kjeldsen 1993, Fischer 2011, pp.157–168). We start with the definition of the moment determinacy of distributions. Let X be a random variable with distribution F (denoted $X \sim F$) and have finite moments $m_k = \mathbf{E}[X^k]$ for all $k = 1, 2, \dots$; namely, the absolute moment $\mu_k = \mathbf{E}[|X|^k] < \infty$ for all positive integers k . If F is uniquely determined by the sequence of its moments $\{m_k\}_{k=1}^{\infty}$, we say that F is moment-determinate (in short, F is M-det, or X is M-det); otherwise, we say that F is moment-indeterminate (F is M-indet, or X is M-indet).

The moment problem is to determine whether or not a given distribution F is M-det. Roughly speaking, there are two kinds of moment problems: Stieltjes (1894) moment problem deals with nonnegative random variables only, while Hamburger (1920, 1921) moment problem treats all random variables taking values in the whole real line.

We recall first two important facts:

Fact A. It is possible that a nonnegative random variable X is M-det in the Stieltjes sense, but M-indet in the Hamburger sense (Akhiezer 1965, p. 240). This happens only for some *discrete* nonnegative random variables with a positive mass at zero (Chihara 1968).

Fact B. If a distribution F is M-indet, then there are infinitely many (i) absolutely continuous distributions, (ii) purely discrete distributions and (iii) singular continuous distributions all having the same moment sequence as F (Berg 1998, Berg and Christensen 1981).

One good reason to study the moment problem was given in Fréchet and Shohat’s (1931) Theorem stated below. Simply speaking, for a given sequence of random variables $X_n \sim F_n, n = 1, 2, \dots$, with finite moments $m_k^{(n)} = \mathbf{E}[X_n^k]$ for all positive integers k , the moment convergence $(\lim_{n \rightarrow \infty} m_k^{(n)} = m_k \forall k)$ does not guarantee the weak convergence of distributions $\{F_n\}_{n=1}^\infty (F_n \xrightarrow{w} F \text{ as } n \rightarrow \infty)$ unless the limiting distribution F is M-det. Therefore, the M-(in)det property is one of the important fundamental properties we have to know about a given distribution.

Fréchet and Shohat’s (1931) Theorem. Let the distribution functions F_n possess finite moments $m_k^{(n)}$ for $k = 1, 2, \dots$ and $n = 1, 2, \dots$. Assume further that the limit $m_k = \lim_{n \rightarrow \infty} m_k^{(n)}$ exists (and is finite) for each k . Then (i) the limits $\{m_k\}_{k=1}^\infty$ are the moment sequence of a distribution function, say F ; (ii) if the limit F given by (i) is M-det, F_n converges to F weakly as $n \rightarrow \infty$.

Necessary and sufficient conditions for the M-det property of distributions exist in the literature (see, e.g., Akhiezer 1961, Shohat and Tamarkin 1943, and Berg et al. 2002), but these conditions are not easily checkable in general. In this survey, we will focus only on the *checkable* M-(in)det criteria for distributions rather than the collection of all specific examples.

In Sections 2 and 3, we review respectively the moment determinacy and moment indeterminacy criteria including Cramér’s condition, Carleman’s condition, Hardy’s condition, Krein’s condition and the growth rate of moments. Some criteria are old, but others are recent. New (direct) proofs for some criteria are provided. To amend some previous proofs in the literature, two lemmas (Lemmas 3 and 4) are given for the first time. We consider in Section 4 the recently formulated Stieltjes classes for M-indet absolutely continuous distributions. Section 5 is devoted to the converses to the previous M-(in)det criteria for distributions. Finally, in Section 6 we review the most recent results about the moment problem for products of independent random variables with different distributions.

2 Checkable criteria for moment determinacy

In this section we consider the checkable criteria for moment determinacy of random variables or distributions. We treat first the Hamburger case because it is more popular than the Stieltjes case. Let $X \sim F$ on the whole real line $\mathbb{R} = (-\infty, \infty)$ with finite moments $m_k = \mathbf{E}[X^k]$ and absolute moment $\mu_k = \mathbf{E}[|X|^k]$ for all positive integers k . For convenience, we define the following statements, in which ‘h’ stands for ‘Hamburger’.

- (h1) $\frac{m_{2(k+1)}}{m_{2k}} = \mathcal{O}((k+1)^2) = \mathcal{O}(k^2)$ as $k \rightarrow \infty$.
- (h2) X has a moment generating function (mgf), i.e., $\mathbf{E}[e^{tX}] < \infty$ for all $t \in (-c, c)$, where $c > 0$ is a constant (Cramér’s condition); equivalently, $\mathbf{E}[e^{t|X|}] < \infty$ for $0 \leq t < c$.
- (h3) $\limsup_{k \rightarrow \infty} \frac{1}{2k} m_{2k}^{1/(2k)} < \infty$.
- (h4) $\limsup_{k \rightarrow \infty} \frac{1}{k} \mu_k^{1/k} < \infty$.
- (h5) $m_{2k} = \mathcal{O}((2k)^{2k})$ as $k \rightarrow \infty$.
- (h6) $m_{2k} \leq c_0^k (2k)!, k = 1, 2, \dots$, for some constant $c_0 > 0$.

- (h7) $C[F] \equiv \sum_{k=1}^{\infty} m_{2k}^{-1/(2k)} = \infty$ (Carleman’s (1926) condition).
- (h8) X is M-det on \mathbb{R} .

Theorem 1 *Under the above settings, if $X \sim F$ on \mathbb{R} satisfies one of the conditions (h1) through (h7), then X is M-det on \mathbb{R} . Moreover, (h1) implies (h2), (h2) through (h6) are equivalent, and (h6) implies (h7). In other words, the following chain of implications holds:*

$$(h1) \implies (h2) \iff (h3) \iff (h4) \iff (h5) \iff (h6) \implies (h7) \implies (h8).$$

We keep the term $k + 1$ in (h1) because it arises naturally in many examples. The first implication in Theorem 1 was given in Stoyanov et al. (2014) recently, while the rest, more or less, are known in the literature. The Carleman quantity $C[F]$ in (h7) is calculated from all even order moments of F . Theorem 1 contains most checkable criteria for moment determinacy in the Hamburger case.

Remark 1 *Some other M-det criteria exist in the literature, but they are seldom used. See, for example, (ha) and (hb) below:*

- (h2) X has a mgf (Cramér’s condition)
- \iff (ha) $\sum_{k=1}^{\infty} \frac{m_{2k}}{(2k)!} x^{2k}$ converges in an interval $|x| < x_0$ (Chow and Teicher 1997, p. 301)
- \implies (hb) $\sum_{k=1}^{\infty} \frac{m_k}{k!} x^k$ converges in an interval $|x| < x_0$ (Billingsley 1995, p. 388)
- \implies (h7) $C[F] = \sum_{k=1}^{\infty} m_{2k}^{-1/(2k)} = \infty$ (Carleman’s condition)
- \implies (h8) X is M-det on \mathbb{R} .

It might look strange that the convergence of subseries in the above (ha) implies the convergence of the whole series in (hb), but remember that the convergence in (ha) holds true for all x in a neighborhood of zero, not just for a fixed x . Billingsley (1995) proved the implication that (hb) \implies (h8) by a version of analytic continuation of characteristic function, but it is easy to see that (hb) also implies (h7) and hence X is M-det on \mathbb{R} .

In Theorem 1, Carleman’s condition (h7) is the weakest checkable condition for X to be M-det on \mathbb{R} . To prove Carleman’s criterion that (h7) implies (h8), we may apply the approach of quasi-analytic functions (Carleman 1926, Koosis 1988), or the approach of Lévy distance (Klebanov and Mkrtchyan 1985). For the latter, we recall the following result.

Klebanov and Mkrtchyan’s (1985) Theorem. Let F and G be two distribution functions on \mathbb{R} and let their first $2n$ moments exist and coincide: $m_k(F) = m_k(G) = m_k$, $k = 1, 2, \dots, 2n$ ($n \geq 2$). Denote the sub-quantity $C_n = \sum_{k=1}^n m_{2k}^{-1/(2k)}$. Then

$$L(F, G) \leq c_2 \frac{\log(1 + C_{n-1})}{(C_{n-1})^{1/4}},$$

where $L(F, G)$ is the Lévy distance and $c_2 = c_2(m_2)$ depends only on m_2 . Therefore, Carleman’s condition (h7) implies that $F = G$ by letting $n \rightarrow \infty$ in Klebanov and Mkrtchyan’s (1985) Theorem. It worths mentioning that Carleman’s condition is sufficient, but not necessary, for a distribution to be M-det. For this, see Heyde (1963b), Stoyanov and Lin (2012, Remarks 5 and 7) or Stoyanov (2013, Section 11).

On the other hand, the statement (h1) in Theorem 1 is the strongest checkable condition for X to be M-det on \mathbb{R} , which means that the growth rate of even order moments is less than or equal to two. The condition (h1) however has its advantage: for some cases,

it is easy to estimate the growth rate (see the example below), because the common factors in the two even order moments, $m_{2(k+1)}$ and m_{2k} , can be cancelled out as n tends to infinity.

Example 1 Consider the double generalized gamma random variable $\xi \sim DGG(\alpha, \beta, \gamma)$ with density function $f(x) = c|x|^{\gamma-1} \exp[-\alpha|x|^\beta]$, $x \in \mathbb{R}$, where $\alpha, \beta, \gamma > 0, f(0) = 0$ if $\gamma \neq 1$, and $c = \beta\alpha^{\gamma/\beta} / (2\Gamma(\gamma/\beta))$ is the norming constant. Then the n th power ξ^n is M-det if $1 \leq n \leq \beta$. To see this known result, we calculate the ratio of even order moments of ξ^n :

$$\frac{E[\xi^{2n(k+1)}]}{E[\xi^{2nk}]} = \frac{\Gamma((\gamma + 2n(k+1))/\beta)}{\alpha^{2n/\beta} \Gamma((\gamma + 2nk)/\beta)} \approx (2n/(\alpha\beta))^{2n/\beta} (k+1)^{2n/\beta} \text{ as } k \rightarrow \infty,$$

by using the approximation of the gamma function: $\Gamma(x) \approx \sqrt{2\pi} x^{x-1/2} e^{-x}$ as $x \rightarrow \infty$. Therefore, ξ^n is M-det if $n \leq \beta$, by the criterion (h1). In fact, for odd integer $n \geq 1$, ξ^n is M-det iff $n \leq \beta$, and for even integer $n \geq 2$, ξ^n is M-det iff $n \leq 2\beta$, regardless of parameter γ . For further results about this distribution and its extensions, see Lin and Huang (1997), Pakes et al. (2001) and Pakes (2014, Theorem 8.3), as well as Examples 3 and 5 below.

Remark 2 We give here a direct proof of the equivalence of statements (h2), (h3) (h5) and (h6). First, for any nonnegative X , we have the equivalence of the following four statements (to be shown later):

- $E[e^{c\sqrt{X}}] < \infty$ for some constant $c > 0$
- iff $m_k \leq c_0^k (2k)!, k = 1, 2, \dots$, for some constant $c_0 > 0$
- iff $\limsup_{k \rightarrow \infty} \frac{1}{k} m_k^{1/(2k)} < \infty$
- iff $m_k = \mathcal{O}(k^{2k})$ as $k \rightarrow \infty$.

Next, consider a general X with $E[e^{t|X|}] < \infty$ for $0 \leq t < c$, namely, $E[e^{t\sqrt{|X|^2}}] < \infty$ for some constant $t > 0$. Then the k th moment of $|X|^2$ is exactly the $2k$ th moment of X and we have immediately the following equivalences (by taking $|X|^2$ as the above nonnegative X):

- (h2) X has a mgf
- iff (h6) $m_{2k} \leq c_0^k (2k)!, k = 1, 2, \dots$, for some constant $c_0 > 0$
- iff $\limsup_{k \rightarrow \infty} \frac{1}{k} m_{2k}^{1/(2k)} < \infty$ (iff (h3) $\limsup_{k \rightarrow \infty} \frac{1}{2k} m_{2k}^{1/(2k)} < \infty$)
- iff $m_{2k} = \mathcal{O}(k^{2k})$ as $k \rightarrow \infty$ (iff (h5) $m_{2k} = \mathcal{O}((2k)^{2k})$ as $k \rightarrow \infty$).

We now present the checkable M-det criteria in the Stieltjes case. Consider $X \sim F$ on $\mathbb{R}_+ = [0, \infty)$ with finite $m_k = \mu_k = E[X^k]$ for all positive integers k , and define the following statements, in which ‘s’ stands for ‘Stieltjes’.

- (s1) $\frac{m_{k+1}}{m_k} = \mathcal{O}((k+1)^2) = \mathcal{O}(k^2)$ as $k \rightarrow \infty$.
- (s2) \sqrt{X} has a mgf (Hardy’s condition), i.e., $E[e^{c\sqrt{X}}] < \infty$ for some constant $c > 0$.
- (s3) $\limsup_{k \rightarrow \infty} \frac{1}{k} m_k^{1/(2k)} < \infty$.
- (s4) $m_k = \mathcal{O}(k^{2k})$ as $k \rightarrow \infty$.
- (s5) $m_k \leq c_0^k (2k)!, k = 1, 2, \dots$, for some constant $c_0 > 0$.
- (s6) $C[F] = \sum_{k=1}^{\infty} m_k^{-1/(2k)} = \infty$ (Carleman’s condition).
- (s7) X is M-det on \mathbb{R}_+ .

Theorem 2 Under the above settings, if $X \sim F$ on \mathbb{R}_+ satisfies one of the conditions (s1) through (s6), then X is M-det on \mathbb{R}_+ . Moreover, (s1) implies (s2), (s2) through (s5) are equivalent, and (s5) implies (s6). In other words, the following chain of implications holds:

$$(s1) \implies (s2) \iff (s3) \iff (s4) \iff (s5) \implies (s6) \implies (s7).$$

The first implication above was given in Lin and Stoyanov (2015). Note that the moment conditions here are in terms of moments of all positive (integer) orders, rather than even order moments as in the Hamburger case. For example, the statement (s1) means that the growth rate of all moments (not only for even order moments) is less than or equal to two. Like Theorem 1, Theorem 2 contains most checkable criteria for moment determinacy in the Stieltjes case. Hardy (1917, 1918) proved that (s2) implies (s7) by two different approaches. Surprisingly, Hardy’s criterion has been ignored for about one century since publication. The following new characteristic properties of (s2) are given in Stoyanov and Lin (2012), from which the equivalence of (s2) through (s5) follows immediately.

Lemma 1 *Let a be a positive constant and X be a nonnegative random variable.*

(i) *If $\mathbf{E}[\exp(cX^a)] < \infty$ for some constant $c > 0$, then $m_k \leq \Gamma(k/a + 1)c_0^k$, $k = 1, 2, \dots$, for some constant $c_0 > 0$.*

(ii) *Conversely, if, in addition, $a \leq 1$, and $m_k \leq \Gamma(k/a + 1)c_0^k$, $k = 1, 2, \dots$, for some constant $c_0 > 0$, then $\mathbf{E}[\exp(cX^a)] < \infty$ for some constant $c > 0$.*

Corollary 1 *Let $a \in (0, 1]$ and $X \geq 0$. Then $\mathbf{E}[\exp(cX^a)] < \infty$ for some constant $c > 0$ iff $m_k \leq \Gamma(k/a + 1)c_0^k$, $k = 1, 2, \dots$, for some constant $c_0 > 0$.*

Lemma 2 *Let a be a positive constant and X be a nonnegative random variable. Then $\limsup_{k \rightarrow \infty} \frac{1}{k} m_k^{a/k} < \infty$ iff $m_k \leq \Gamma(k/a + 1)c_0^k$, $k = 1, 2, \dots$, for some constant $c_0 > 0$.*

Corollary 2 *Let $a \in (0, 1]$ and $X \geq 0$. Then $\mathbf{E}[\exp(cX^a)] < \infty$ for some constant $c > 0$ iff $\limsup_{k \rightarrow \infty} \frac{1}{k} m_k^{a/k} < \infty$.*

We mention that for any nonnegative X , its mgf exists iff $\limsup_{k \rightarrow \infty} \frac{1}{k} m_k^{1/k} < \infty$ due to Corollary 2. This in turn implies the equivalence of (h2) and (h4) in Theorem 1 for the Hamburger case. More general results in terms of absolute moments are given below. For easy comparison, some statements are repeated here.

Equivalence Theorem A (Hamburger case). Let $p \geq 1$ be a constant and the random variable $X \sim F$ on \mathbb{R} . Denote $m_k = \mathbf{E}[X^k]$ for integer $k \geq 1$ and let $\mu_\ell = \mathbf{E}[|X|^\ell] < \infty$ for all $\ell > 0$. Then the following statements are equivalent:

- (a) X satisfies Cramér’s condition, namely, the moment generating function of X exists.
- (b) $\mu_k \leq c_0^k k!$, $k = 1, 2, \dots$, for some constant $c_0 > 0$.
- (c) $\mu_{pk} \leq c_0^k \Gamma(pk + 1)$, $k = 1, 2, \dots$, for some constant $c_0 > 0$.
- (d) $\limsup_{k \rightarrow \infty} \frac{1}{pk} \mu_{pk}^{1/(pk)} < \infty$.
- (e) $m_{2k} \leq c_0^k (2k)!$, $k = 1, 2, \dots$, for some constant $c_0 > 0$.
- (f) $\limsup_{k \rightarrow \infty} \frac{1}{2k} m_{2k}^{1/(2k)} < \infty$.

Proof The equivalence of (a), (b), (e) and (f) was given in Theorem 1. To prove the remaining relations, denote $X_* = |X|$ and write $Y_p = X_*^p$ and $v_{k,p} = \mathbf{E} \left[Y_p^k \right] = \mu_{pk}$. Then note further that $\mathbf{E} \left[e^{cX_*} \right] = \mathbf{E} \left[e^{c(Y_p)^{1/p}} \right] < \infty$ for some constant $c > 0$ iff $v_{k,p} \leq c_0^k \Gamma(pk + 1)$, $k = 1, 2, \dots$, for some constant $c_0 > 0$ (by taking $a = 1/p$ and $X = Y_p$ in Lemma 1) iff (c) holds true. On the other hand, $v_{k,p} \leq c_0^k \Gamma(pk + 1)$, $k = 1, 2, \dots$, for some

constant $c_0 > 0$ iff $\limsup_{k \rightarrow \infty} \frac{1}{k} \nu_{k,p}^{1/(pk)} < \infty$ (by Lemma 2) iff (d) holds true. The proof is complete. \square

The above statements (e) and (f) are special cases of (c) and (d) with $p = 2$, respectively. Similarly, we give the following equivalence theorem without proof for Stieltjes case.

Equivalence Theorem B (Stieltjes case). Let $p \geq 1$ be a constant. Let the random variable $0 \leq X \sim F$ on \mathbb{R}_+ with finite $m_k = \mu_k = E[X^k]$ for all integers $k \geq 1$. Then the following statements are equivalent:

- (a) X satisfies Hardy’s condition, namely, the moment generating function of \sqrt{X} exists.
- (b) $\mu_k \leq c_0^k (2k)!, k = 1, 2, \dots$, for some constant $c_0 > 0$.
- (c) $\mu_{pk} \leq c_0^k \Gamma(2pk + 1), k = 1, 2, \dots$, for some constant $c_0 > 0$.
- (d) $\limsup_{k \rightarrow \infty} \frac{1}{pk} \mu_{pk}^{1/(2pk)} < \infty$.
- (e) $\limsup_{k \rightarrow \infty} \frac{1}{k} \mu_k^{1/(2k)} < \infty$.

3 Checkable criteria for moment indeterminacy

In this section we consider the checkable criteria for moment indeterminacy. Krein (1945) proved the following remarkable criterion in the Hamburger case.

Krein’s Theorem. Let $X \sim F$ on \mathbb{R} have a positive density function f and finite moments of all positive orders. Assume further that the Lebesgue logarithmic integral

$$K[f] \equiv \int_{-\infty}^{\infty} \frac{-\log f(x)}{1+x^2} dx < \infty. \tag{1}$$

Then F is M-indet on \mathbb{R} .

We call the logarithmic integral $K[f]$ in (1) the Krein integral for the density f . Graffi and Grecchi (1978) as well as Slud (1993) proved independently the counterpart of Krein’s Theorem for the Stieltjes case by the method of symmetrization of a distribution on \mathbb{R}_+ . To give a constructive and complete proof, we however need Lemma 3 below (see, e.g., Lin 1997, Theorem 3, and Rao et al. 2009, Remark 8).

Graffi, Grecchi and Slud’s Theorem. Let $X \sim F$ on \mathbb{R}_+ have a positive density function f and finite moments of all positive orders. Assume further that the integral

$$K[f] = \int_0^{\infty} \frac{-\log f(x^2)}{1+x^2} dx < \infty. \tag{2}$$

Then F is M-indet on \mathbb{R}_+ and hence M-indet on \mathbb{R} .

Lemma 3 *Let Y have a symmetric distribution G with density g and finite moments of all positive orders. If the integral*

$$K[g] = \int_{-\infty}^{\infty} \frac{-\log g(x)}{1+x^2} dx < \infty,$$

then there exists a symmetric distribution $G_ \neq G$ having the same moment sequence as G .*

Proof By the assumptions of the lemma, there exists a complex-valued function ϕ such that $|\phi| = g$ (in the sense of almost everywhere) and

$$\int_{-\infty}^{\infty} \phi(x) e^{itx} dx = 0, \quad t \geq 0$$

(see the proof of Theorem 1 in Lin 1997 for details, and Garnett 1981, p.66, for the construction of ϕ). The last equality implies that

$$\int_{-\infty}^{\infty} x^k \phi(x) e^{itx} dx = 0, \quad t \geq 0, \quad k = 0, 1, 2, \dots$$

In particular,

$$\int_{-\infty}^{\infty} x^k \phi(x) dx = 0, \quad k = 0, 1, 2, \dots$$

Let $\phi = \phi_1 + i\phi_2$, then both ϕ_j are real and $|\phi_j| \leq g$. We have

$$\int_{-\infty}^{\infty} x^k \phi_j(x) dx = 0, \quad j = 1, 2, \quad k = 0, 1, 2, \dots$$

We split the rest of the proof into three cases:

(i) $\phi_1 \neq 0, \phi_2 = 0$, (ii) $\phi_1 = 0, \phi_2 \neq 0$, and (iii) $\phi_1 \neq 0, \phi_2 \neq 0$.

(i) If ϕ_1 is odd, then for each $t > 0$, the function $\phi_*(x) := \phi_1(x) \sin(tx)$ is even and

$$\int_{-\infty}^{\infty} x^k \phi_*(x) dx = 0, \quad k = 0, 1, 2, \dots$$

Take $g_* = g + \phi_* \neq g$. Then $g_* \geq 0$ is even and has the same moment sequence as g . On the other hand, if ϕ_1 is not odd, then let first $\ell(x) = \frac{1}{2}[\phi_1(x) + \phi_1(-x)]$ which is even and satisfies

$$\int_{-\infty}^{\infty} x^k \ell(x) dx = 0, \quad k = 0, 1, 2, \dots$$

Next, take $g_* = g + \ell \neq g$, which has the same moment sequence as g .

(ii) The proof of this case is similar to that of case (i).

(iii) If one ϕ_j is not odd, then it is done as in (i) (by taking $\ell(x) = \frac{1}{2}[\phi_j(x) + \phi_j(-x)]$ and $g_* = g + \ell$). Suppose now that both ϕ_j are odd, then, by the definition of ϕ , we further have $\int_{-\infty}^{\infty} \phi(x) e^{itx} dx = 0 \forall t \in \mathbb{R}$. Let $t > 0$ be fixed and define the function $\psi(x) = \phi_1(x) \sin(tx) + \phi_2(x) \cos(tx)$ (the imaginary part of $\phi(x) e^{itx}$), then

$$\int_{-\infty}^{\infty} x^k \psi(x) dx = 0, \quad \int_{-\infty}^{\infty} x^k \psi(-x) dx = 0, \quad k = 0, 1, 2, \dots$$

Take $m(x) = \frac{1}{2}[\psi(x) + \psi(-x)] = \phi_1(x) \sin(tx) \neq 0$, which is even and satisfies

$$\int_{-\infty}^{\infty} x^k m(x) dx = 0, \quad k = 0, 1, 2, \dots$$

We have $g_* = g + m \neq g$, which is nonnegative and has the same moment sequence as g . The proof is complete. \square

It should be noted that in the logarithmic integral (2), the argument of the density function f is x^2 rather than x as in (1). Recently, Pedersen (1998) improved Krein's Theorem by the concept of positive lower uniform density sets and proved that it suffices to calculate the Krein integral over the two-sided tail of the density function (instead of the whole line).

Theorem 3 (Pedersen 1998). *Let $X \sim F$ on \mathbb{R} have a density function f and finite moments of all positive orders. Assume further that the integral*

$$K[f] = \int_{|x| \geq c} \frac{-\log f(x)}{1+x^2} dx < \infty \text{ for some } c \geq 0. \tag{3}$$

Then X is M -indeterminate on \mathbb{R} .

See also Hörfelt (2005) for Theorem 3 with a different proof (provided by H.L. Pedersen). Pedersen (1998) also showed by giving an example that Krein’s condition (1) is sufficient, but not necessary, for a distribution to be M-indet. This corrected the statement (2) in Leipnik (1981) about Krein’s condition. On the other hand, Pakes (2001) and Hörfelt (2005) pointed out the counterpart of Pedersen’s Theorem for the Stieltjes case. To prove this result, we need Lemma 4 below.

Theorem 4 (Pakes 2001, Hörfelt 2005). *Let $X \sim F$ on \mathbb{R}_+ have a density function f and finite moments of all positive orders. Assume further that the integral*

$$K[f] = \int_{x \geq c} \frac{-\log f(x^2)}{1+x^2} dx < \infty \text{ for some } c \geq 0. \tag{4}$$

Then X is M-indet on \mathbb{R}_+ and hence M-indet on \mathbb{R} .

Lemma 4 *Let $0 \leq X \sim F$ with density f and finite moments of all positive orders. Let $Y \sim G$ with density g be the symmetrization of \sqrt{X} . If for some $c \geq 0$,*

$$K[g] = \int_{|x| \geq c} \frac{-\log g(x)}{1+x^2} dx < \infty,$$

then X is M-indet on \mathbb{R}_+ .

Proof Under the condition on the logarithmic integral of g , Pedersen (1998, Theorem 2.2) proved that the set of polynomials is not dense in $L^1(\mathbb{R}, g(x)dx)$. This implies that the set of polynomials is not dense in $L^2(\mathbb{R}, g(x)dx)$ either (see, e.g., Berg and Christensen 1981, or Goffman and Pedrick 2002, p. 162). Then proceeding along the same lines as in the proof of Corollary 1 in Slud (1993), we conclude that the set of polynomials is not dense in $L^2(\mathbb{R}, f(x)dx)$. Therefore, X is M-indet on \mathbb{R} , which in turn implies that X is M-indet on \mathbb{R}_+ due to Chihara’s (1968) result in Fact A above. The proof is complete. \square

Conversely, once we prove Theorem 4, we can extend Lemma 4 as follows.

Lemma 4*. *If $X \sim F$ on \mathbb{R} satisfies the conditions in Theorem 3, then X^2 is M-indet.*

Proof Apply Theorem 4 above and Pakes et al.’s (2001) Theorem 3(i): If $X \sim F$ on \mathbb{R} satisfies condition (3), then the Krein integral $K[f_2]$ in (4) of X^2 is finite, where f_2 is the density of X^2 . \square

For the M-det case, a trivial analogue of Lemma 4* is the following.

Lemma 4**. *If $X \sim F$ on \mathbb{R} satisfies Carleman’s condition (h7), then X^2 satisfies Carleman’s condition (s6) and is M-det on \mathbb{R}_+ .*

For simplicity, all the conditions (1) through (4) are called Krein’s condition. For illustration of how to use Krein’s and Hardy’s criteria, we now recover Berg’s (1988) results using these powerful criteria (see also Prohorov and Rozanov 1969, p. 167, Pakes and Khattree 1992, Lin and Huang 1997, and Stoyanov 2000).

Example 2 *Let X have a normal distribution and $\alpha > 0$. Then*

(i) the odd power X^{2n+1} is M-indet if $n \geq 1$, and (ii) $|X|^\alpha$ is M-det iff $\alpha \leq 4$.

Without loss of generality, we assume that X has a density $f(x) = \frac{1}{\sqrt{\pi}} \exp(-x^2)$, $x \in \mathbb{R}$, namely, $\sqrt{2}X$ has a standard normal distribution. We discuss these results in three steps.

(I) Berg (1988) proved the moment indeterminacy of distributions by giving examples. For part (i), he calculated first the density of X^{2n+1} :

$$f_n(x) = \frac{1}{(2n + 1)\sqrt{\pi}} |x|^{-2n/(2n+1)} \exp\left(-|x|^{2/(2n+1)}\right), \quad x \in \mathbb{R},$$

and then constructed the density function

$$\begin{aligned} f_{r,n}(x) &= f_n(x) \left\{ 1 + r \left[\cos\left(\beta_n |x|^{2/(2n+1)}\right) - \gamma_n \sin\left(\beta_n |x|^{2/(2n+1)}\right) \right] \right\} \\ &\equiv f_n(x) \{1 + r p_n(x)\}, \quad x \in \mathbb{R}, \end{aligned}$$

where $|r| \leq \sin \frac{\pi}{2(2n+1)}$, $\beta_n = \tan \frac{\pi}{2n+1}$ and $\gamma_n = \cot \frac{\pi}{2(2n+1)}$. It is seen that $f_{r,n} \neq f_n$ if $r \neq 0$ and $n \geq 1$, but $f_{r,n}$ and f_n have the same moment sequence because the product of the density f_n and the function p_n defined above has vanishing moments by a tedious calculation:

$$\int_{-\infty}^{\infty} x^k f_n(x) p_n(x) dx = 0, \quad k = 0, 1, 2, \dots$$

This proves part (i). Alternatively, we note however that the Krein integral

$$K[f_n] = \int_{-\infty}^{\infty} \frac{-\log f_n(x)}{1 + x^2} dx = C + \int_{-\infty}^{\infty} \frac{|x|^{2/(2n+1)}}{1 + x^2} dx < \infty \quad (\text{if } n \geq 1),$$

which implies by Krein's Theorem that the odd power X^{2n+1} is M-indet if $n \geq 1$.

(II) For part (ii), the density of $|X|^\alpha$ is

$$f_\alpha(x) = \frac{2}{\alpha\sqrt{\pi}} x^{1/\alpha-1} \exp(-x^{2/\alpha}), \quad x \geq 0.$$

If $\alpha > 4$, Berg constructed again the density function

$$f_{r,\alpha}(x) = f_\alpha(x) \left\{ 1 + r \left[\cos\left(\beta_\alpha x^{2/\alpha}\right) - \gamma_\alpha \sin\left(\beta_\alpha x^{2/\alpha}\right) \right] \right\} \equiv f_\alpha(x) \{1 + r p_\alpha(x)\}, \quad x \geq 0,$$

where $|r| \leq \sin(\pi/\alpha)$, $\beta_\alpha = \tan(2\pi/\alpha)$ and $\gamma_\alpha = \cot(\pi/\alpha)$. Then $f_{r,\alpha} \neq f_\alpha$ if $r \neq 0$ and $\alpha > 4$, but $f_{r,\alpha}$ and f_α have the same moment sequence because

$$\int_0^\infty x^k f_\alpha(x) p_\alpha(x) dx = 0, \quad k = 0, 1, 2, \dots$$

Therefore, $|X|^\alpha$ is M-indet if $\alpha > 4$. Again, we see that the Krein integral (in Stieltjes case)

$$K[f_\alpha] = \int_0^\infty \frac{-\log f_\alpha(x^2)}{1 + x^2} dx = C + \int_0^\infty \frac{x^{4/\alpha}}{1 + x^2} dx < \infty \quad (\text{if } \alpha > 4).$$

So the required result follows immediately from Krein's criterion (4).

(III) For the rest of part (ii), Berg calculated the k th moment of $|X|^\alpha$:

$$m_{\alpha,k} = \int_0^\infty x^k f_\alpha(x) dx = \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{\alpha k + 1}{2}\right), \quad k = 0, 1, 2, \dots$$

By Stirling's formula, $m_{\alpha,k}^{1/k} \approx ck^{\alpha/2}$ as $k \rightarrow \infty$, and hence the Carleman quantity (in Stieltjes case) is equal to

$$C[f_\alpha] = \sum_{k=1}^{\infty} m_{\alpha,k}^{-1/(2k)} = \infty \quad (\text{if } \alpha \leq 4).$$

This proves the necessary part of (ii). Instead, we note that if $\alpha \in (0, 2]$, the mgf of $|X|^\alpha$ exists by its density function above, and hence $|X|^{2\alpha}$ is M-det by Hardy's criterion.

There are some ramifications of the moment problem for normal random variables. For example, Slud (1993) investigated the moment problem for polynomial forms in normal random variables, while Hörfelt (2005) studied the moment problem for some Wiener functionals which extend Berg’s results in Example 2. Besides, Lin and Huang (1997) treated the double generalized Gamma (DGG) distribution as an extension of the normal one and found the necessary and sufficient conditions for powers of DGG random variable to be M-det.

4 Stieltjes classes for M-indet distributions

Stieltjes (1894) observed that some positive measures, e.g., $\mu(dx) = e^{-x^{1/4}} dx$ or $x^{n-\log x} dx$ (n is an integer), are not unique by moments. This might be the starting point of T. J. Stieltjes to study the moment problem (see Kjeldsen 1993). It was C. C. Heyde who first presented this phenomenon in probability language and proved in 1963 that the lognormal distribution is M-indet by giving the example described next. Consider the standard lognormal density

$$f(x) = \frac{1}{\sqrt{2\pi}} x^{-1} \exp\left[-\frac{1}{2}(\log x)^2\right], \quad x > 0,$$

with moment sequence $\{\exp(k^2/2)\}_{k=1}^\infty$. Then, for each $\varepsilon \in [-1, 1]$,

$$\int_0^\infty f(x)[1 + \varepsilon \sin(2\pi \log x)] x^k dx = \int_0^\infty f(x)x^k dx \quad \forall k = 0, 1, 2, \dots$$

because the product of the density $f(x)$ and the function $\sin(2\pi \log x)$ has vanishing moments:

$$\int_0^\infty f(x)[\sin(2\pi \log x)] x^k dx = \frac{e^{k^2/2}}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-x^2/2} \sin(2\pi x) dx = 0 \quad \forall k = 0, 1, 2, \dots$$

There are many other distributions having the same moment sequence as the above lognormal with mean \sqrt{e} , including (i) the ones with density f satisfying the functional equation: $f(qx) = q^{-1/2}xf(x)$, where $q = 1/e \in (0, 1)$ (see, e.g., López-García 2011, Theorem 1), or, more generally, (ii) the distributions F satisfying $F(x) = e^{-1/2} \int_0^{ex} u dF(u)$, $x \geq 0$ (Pakes 1996, Section 3). The latter showed that each such F corresponds to a finite measure in the interval $(1/e, 1]$ and vice versa. Hence the cardinality of the set of all solutions to the functional equation is $\aleph_2 = 2^{\mathbb{R}}$. All these distributions are called the solutions to the lognormal moment problem (see also Chihara 1970, Leipnik 1982, Pakes 2007 and Christiansen 2003).

Recently, Stoyanov (2004) formulated Stieltjes classes for M-indet absolutely continuous distributions as follows. Let $X \sim F$ have an M-indet distribution on \mathbb{R} with density f . A Stieltjes class \mathcal{S} for F is defined by

$$\mathcal{S} = \mathcal{S}(f, p) = \{f_\varepsilon : f_\varepsilon(x) = f(x)[1 + \varepsilon p(x)], \quad x \in \mathbb{R}, \quad \varepsilon \in [-1, 1]\},$$

where p is a measurable function (called a perturbation function) such that $|p(x)| \leq 1$ and

$$\int_{-\infty}^\infty f(x)p(x)x^k dx = 0, \quad k = 0, 1, 2, \dots$$

We note that for a given M-indet distribution, the choice of the function p might not be unique. Besides the previous lognormal and normal results of Heyde (1963a) and Berg (1988), some other perturbation functions are given below:

1. If X has a generalized Weibull density $f(x) = \frac{1}{24} \exp(-x^{1/4})$, $x > 0$, then $p(x) = \sin(x^{1/4})$, $x > 0$ (Stieltjes 1894, Serfling 1980).
2. If X has a density function $f(x) = cx^{-\log x}$, $x > 0$, where c is a norming constant, then we choose $p(x) = \sin(2\pi \log x)$, $x > 0$ (Stieltjes 1894).
3. If X has a gamma density with parameter $\alpha > 0$, then X^β is M-indet provided $\beta > \max\{2, 2\alpha\}$, and we can choose

$$p(x) = \sin(\alpha\pi/\beta) [\cos(\tan(\pi/\beta)x^{1/\beta}) - \cot(\alpha\pi/\beta) \sin(\tan(\pi/\beta)x^{1/\beta})], \quad x > 0$$

(Targhetta 1990).

4. If X has a density function $f(x) = c \exp(-\alpha|x|^\rho)$, $x \in \mathbb{R}$, where $\alpha > 0$, $\rho \in (0, 1)$ and c is a norming constant, then we choose $p(x) = \cos(\alpha|x|^\rho)$, $x \in \mathbb{R}$ (Prohorov and Rozanov 1969, p. 167).
5. For the log-skew-normal distribution with parameter $\lambda > 0$, we choose the perturbation function

$$p(x) = \frac{\ell(x-1) \sin[\pi \log(x-1)]}{\ell(x) \Phi(\lambda \log x)}, \quad \text{if } x > 1,$$

and $p(x) = 0$, otherwise, where ℓ is the density of standard lognormal LN(0,1) and Φ is the standard normal distribution (Lin and Stoyanov 2009).

Several systematic approaches for constructing Stieltjes classes are available. For example, for any M-indet distribution F on $(0, \infty)$ with density f bounded from below as $f(x) \geq A \exp(-\alpha x^\beta)$, $x > 0$, where $A > 0$, $\alpha > 0$ and $\beta \in (0, 1/2)$ are constants, we find first a complex-valued function g satisfying

- (i) g is analytic in $\mathbb{C}_+ \setminus \{0\}$, where $\mathbb{C}_+ = \{z : \text{Im } z \geq 0\}$ is the upper half-plane, and
- (ii) $g(x) \in \mathbb{R}$, $x > 0$, and $|g(z)| \leq A \exp(-\alpha|z|^\beta)$, $z \in \mathbb{C}_+ \setminus \{0\}$.

Then choose the perturbation function $p(x) = [\text{Im } g(-x)]/f(x)$, $x > 0$ (Ostrowska 2014).

On the other hand, given any Stieltjes class $\mathcal{S}(f, p)$ defined above and a positive random variable V with distribution H and finite moments of all positive orders, we can construct a new Stieltjes class $\mathcal{S}(f^*, p^*)$ by random scaling: $Y_\varepsilon := VX_\varepsilon$, $\varepsilon \in [-1, 1]$, where the random variable X_ε has density f_ε , V is independent of X_ε , $f^* = f_0^*$ is the density of $Y_0 = VX$, and the perturbation function p^* satisfies $f^*(x)p^*(x) = \int_0^\infty v^{-1}f(x/v)p(x/v)dH(v)$, $x \in \mathbb{R}$ (Pakes 2007, Section 5).

For more perturbation functions, see Stoyanov (2004), Stoyanov and Tolmatz (2004, 2005), Ostrowska and Stoyanov (2005), Gómez and López-García (2007), Penson et al. (2010), Wang (2012), Kleiber (2013, 2014) and Ostrowska (2016).

5 Converse criteria

In this section we present some converses to the previous M-(in)det criteria. Recall that for the Stieltjes case, if (s1) above holds true, i.e., $m_{k+1}/m_k = \mathcal{O}((k+1)^2)$ as $k \rightarrow \infty$, then X is M-det on \mathbb{R}_+ . One might guess that if the moments $\{m_k\}_{k=1}^\infty$ grow faster, then X becomes M-indet. This is true under one more condition defined below (see Lin 1997, Stirzaker 2015, p. 223, Kopanov and Stoyanov 2017, or Stoyanov and Kopanov 2017).

Condition L: Suppose, in the Stieltjes case, that f is a density function on \mathbb{R}_+ such that for some fixed $x_0 \geq 0$, f is strictly positive and differentiable on (x_0, ∞) and

$$L_f(x) := -\frac{xf'(x)}{f(x)} \nearrow \infty \text{ as } x_0 < x \rightarrow \infty.$$

In the Hamburger case we require the density $f(x), x \in \mathbb{R}$, to be symmetric about zero.

Theorem 5 *Let X be a nonnegative random variable with distribution F and let its moments grow fast in the sense that $m_{k+1}/m_k \geq c(k+1)^{2+\varepsilon}$ for all large k , where c and ε are positive constants. Assume further that X has a density function f which satisfies Condition L. Then X is M-indet.*

Note that in the above theorem, X is M-indet on \mathbb{R}_+ iff it is M-indet on \mathbb{R} because X has a density. For the Hamburger case, we have the following.

Theorem 6 *Suppose the moments of $X \sim F$ on \mathbb{R} grow fast in the sense that $m_{2(k+1)}/m_{2k} \geq c(k+1)^{2+\varepsilon}$ for all large k , where c and ε are positive constants. Assume further that X has a density function f which is symmetric about zero and satisfies Condition L. Then X satisfies Krein's condition, and hence both X and X^2 are M-indet.*

The crucial point in the proofs of Theorems 5 and 6 is to prove that the Krein integral $K[f] < \infty$ by Condition L and the moment condition. The M-indet property of X^2 in Theorem 6 is due to Lemma 4* and Fact A above. Similarly, we have the following results for other criteria (s4) and (h5) (see Lin and Stoyanov 2015, 2016, and Stoyanov et al. 2014).

Theorem 7 (Stieltjes case). *Let $X \sim F$ on \mathbb{R}_+ and let its moments grow fast in the sense that $m_k \geq ck^{(2+\varepsilon)k}, k = 1, 2, \dots$, for some positive constants c and ε . Assume further that X has a density function f which satisfies Condition L. Then X is M-indet.*

Theorem 8 (Hamburger case). *Suppose the moments of $X \sim F$ grow fast in the sense that $m_{2k} \geq c(2k)^{(2+\varepsilon)k}, k = 1, 2, \dots$, for some positive constants c and ε . Assume further that X has a density function f which is symmetric about zero and satisfies Condition L. Then X satisfies Krein's condition, and hence both X and X^2 are M-indet.*

Note that Condition L also applies to converse M-indet criteria. Actually, this is the original purpose of the condition, under which $K[f] = \infty$ implies $C[F] = \infty$ (Lin 1997). The M-det property of X^2 in the next result is due to Lemma 4** and Fact A above.

Theorem 9 *In Theorem 3 (Hamburger case), if the Krein integral $K[f] = \infty$ and if f satisfies Condition L, then X satisfies Carleman's condition, and hence both X and X^2 are M-det.*

Theorem 10 *In Theorem 4 (Stieltjes case), if the Krein integral $K[f] = \infty$ and if f satisfies Condition L, then X satisfies Carleman's condition and is M-det.*

Remark 3 *In view of Theorems 9 and 10 above, we know that in the class of absolutely continuous distributions with density functions satisfying Condition L, Krein's condition ((3) or (4)) becomes necessary and sufficient for a distribution to be M-indet.*

Remark 4 *In the above converse results, it is possible to replace Condition L by other slightly weaker conditions (mathematically) like those in Pakes (2001) and Gut (2002), but as mentioned before, we focus only on the checkable conditions in this survey. Interestingly, Condition L is closely related to a useful concept in reliability theory. More precisely, if a nonnegative random variable X with density $F' = f$ satisfies Condition L on \mathbb{R}_+ with $x_0 = 0$, then it has an increasing generalized failure rate (by Theorem 1 in Lariviere 2006), namely, the product function $xf(x)/\bar{F}(x)$ (of x and the failure rate) increases in x .*

In addition to the previous problems for normal distributions, we mention here some more variants for general cases, but we are not going to pursuit all the moment problems. To solve these problems, we need to derive new auxiliary tools case by case (like Lemma 5 below). Lin and Stoyanov (2002) and Gut (2003) studied the moment problem for random sums of independently identically distributed (i.i.d.) random variables. Stoyanov et al. (2014) and Lin and Stoyanov (2015) investigated the moment problem for products of i.i.d. random variables. In the next section we review the recent results about products of independent random variables with *different* distributions; for details, see Lin and Stoyanov (2016).

6 Moment problem for products of random variables

Products of random variables occur naturally in stochastic modelling of complex random phenomena in areas such as statistical physics, quantum theory, communication theory, reliability theory and financial modelling; especially in modern communications (see, e.g., Chen et al. 2012, Springer 1979, and Galambos and Simonelli 2004). We split the problem in question into three cases: (a) products of nonnegative random variables, (b) products of random variables taking values in \mathbb{R} , and (c) the mixed case. Moreover, all random variables considered have finite moments of all positive orders.

6.1 Products of nonnegative random variables

The M-det result (Theorem 11 below) is an easy consequence of Theorem 2, while the hard part is the M-indet result (Theorem 12) whose proof needs a delicate analysis.

Theorem 11 *Let ξ_1, \dots, ξ_n be independent nonnegative random variables and let the moments $m_{i,k} = \mathbf{E} \left[\xi_i^k \right]$, $i = 1, \dots, n$, satisfy the conditions:*

$$m_{i,k} = \mathcal{O}(k^{a_i k}) \text{ as } k \rightarrow \infty, \text{ for } i = 1, \dots, n,$$

where a_1, \dots, a_n are positive constants. If the parameters a_i are such that $\sum_{i=1}^n a_i \leq 2$, then the product $Z_n = \prod_{i=1}^n \xi_i$ satisfies Hardy's condition and is M-det.

Theorem 12 *Consider n independent nonnegative random variables, $\xi_i \sim F_i$, $i = 1, 2, \dots, n$, where $n \geq 2$. Suppose that each F_i is absolutely continuous and has a positive density f_i on $(0, \infty)$ and that the following conditions are satisfied:*

- (i) *At least one of the densities $f_1(x), \dots, f_n(x)$ is decreasing in $[x_0, \infty)$, where $x_0 \geq 1$ is a constant.*
- (ii) *For each $i = 1, 2, \dots, n$, there exists a constant $A_i > 0$ such that the density f_i and the tail function $\bar{F}_i(x) = 1 - F_i(x) = \Pr(\xi_i > x)$ together satisfy the relation*

$$f_i(x)/\bar{F}_i(x) \geq A_i/x \text{ for } x \geq x_0, \tag{5}$$

and there exist constants $B_i > 0$, $\alpha_i > 0$, $\beta_i > 0$ and real γ_i such that

$$\bar{F}_i(x) \geq B_i x^{\gamma_i} \exp(-\alpha_i x^{\beta_i}) \text{ for } x \geq x_0. \tag{6}$$

If, in addition to the above, $\sum_{i=1}^n 1/\beta_i > 2$, then the product $Z_n = \prod_{i=1}^n \xi_i$ is M -indet.

Let us explain the above conditions. In terms of reliability language, the failure rate in (5) and the survival function in (6) cannot approach zero too quickly. In other words, (5) and (6) control the tail (decreasing) behavior of the related distributions in some sense. There are three key steps in the proof of Theorem 12: (i) represent the density function of the product Z_n in multiple integral form, (ii) estimate the lower bound of the density function by truncating the two tails of this integral, and (iii) apply Krein’s criterion for the Stieltjes case. For estimation in the step (ii), we need the following auxiliary tool which can be proved using integration by parts.

Lemma 5 *Let F be a distribution on \mathbb{R} such that (i) it has density f on the subset $[a, ra]$, where $a > 0$ and $r > 1$, and (ii) for some constant $A > 0$, $f(x)/\bar{F}(x) \geq A/x$ on $[a, ra]$. Then*

$$\int_a^{ra} \frac{f(x)}{x} dx \geq \left(1 - \frac{1}{r}\right) \frac{A}{1+A} \frac{\bar{F}(a)}{a}.$$

Example 3 *For illustration of how to use Theorems 11 and 12, consider the generalized gamma distributions. We say that $\xi \sim GG(\alpha, \beta, \gamma)$ if its density is of the form*

$$f(x) = c x^{\gamma-1} \exp(-\alpha x^\beta), \quad x \geq 0.$$

Here $\alpha, \beta, \gamma > 0$, $f(0) = 0$ if $\gamma \neq 1$, and $c = \beta \alpha^{\gamma/\beta} / \Gamma(\gamma/\beta)$ is the norming constant. Then we have the following characterization result (see also Pakes 2014 for a much more general result with different proof):

Suppose that ξ_1, \dots, ξ_n are n independent random variables and let $\xi_i \sim GG(\alpha_i, \beta_i, \gamma_i)$, $i = 1, \dots, n$. Then the product $Z_n = \prod_{i=1}^n \xi_i$ is M -det iff $\sum_{i=1}^n 1/\beta_i \leq 2$.

Example 4 *Consider the class of inverse Gaussian distributions. We say that $\xi \sim IG(\mu, \lambda)$ if its density is of the form*

$$f(x) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\left[-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right], \quad x > 0,$$

where $\mu, \lambda > 0$ and $f(0) = 0$. It can be shown that the product of two independent random variables is M -det if each one is exponential or inverse Gaussian, while the product of three such random variables is M -indet. For the powers of such random variables and others, see, e.g., Lin and Huang (1997), Stoyanov (1999), Pakes et al. (2001), Stoyanov et al. (2014) and Lin and Stoyanov (2015). Here are some recent results.

Let $\xi_1 \sim IG(\mu_1, \lambda_1)$, $\xi_2 \sim IG(\mu_2, \lambda_2)$ and $\eta \sim Exp(1) = GG(1, 1, 1)$ be three independent random variables. Then both the products $\xi_1 \eta$ and $\xi_1 \xi_2$ are M -det, while $\xi_1 \xi_2 \eta$ is M -indet.

6.2 Products of random variables taking values in \mathbb{R}

For this Hamburger case, we have the counterparts of Theorems 11 and 12 as follows. In the proof of Theorem 14, the symmetric condition on the densities plays a crucial role.

Theorem 13 Let ξ_1, \dots, ξ_n be independent random variables and let the even order moments $m_{i,2k} = \mathbf{E} \left[\xi_i^{2k} \right]$, $i = 1, \dots, n$, satisfy the conditions:

$$m_{i,2k} = \mathcal{O}((2k)^{2a_i k}) \text{ as } k \rightarrow \infty, \text{ for } i = 1, \dots, n,$$

where a_1, \dots, a_n are positive constants. If the parameters a_i are together such that $\sum_{i=1}^n a_i \leq 1$, then the product $Z_n = \prod_{i=1}^n \xi_i$ satisfies Cramér’s condition and is M-det.

Theorem 14 Consider n independent random variables $\xi_i \sim F_i$, $i = 1, \dots, n$, where $n \geq 2$. Suppose each F_i has a positive density f_i on \mathbb{R} and symmetric about zero. Assume further that

(i) at least one of the densities $f_1(x), \dots, f_n(x)$ is decreasing in $[x_0, \infty)$, where $x_0 \geq 1$ is a constant, and

(ii) for all i , f_i/\bar{F}_i satisfies the condition (5): $f_i(x)/\bar{F}_i(x) \geq A_i/x$ for $x \geq x_0$, and \bar{F}_i satisfies the condition (6): $\bar{F}_i(x) \geq B_i x^{\gamma_i} \exp(-\alpha_i x^{\beta_i})$ for $x \geq x_0$.

If, in addition to the above, $\sum_{i=1}^n 1/\beta_i > 1$, then the product $Z_n = \prod_{i=1}^n \xi_i$ satisfies Krein’s condition, and hence both Z_n and Z_n^2 are M-indet.

Example 5 Applying Theorems 13 and 14 to the product of double generalized gamma random variables $\xi \sim DGG(\alpha, \beta, \gamma)$, defined above, yields the following interesting result:

Suppose that ξ_1, \dots, ξ_n are n independent random variables, and let $\xi_i \sim DGG(\alpha_i, \beta_i, \gamma_i)$, $i = 1, 2, \dots, n$. Then the product $Z_n = \prod_{i=1}^n \xi_i$ is M-det iff $\sum_{i=1}^n 1/\beta_i \leq 1$ iff Z_n^2 is M-det.

6.3 The mixed case

Finally, we consider the products of both types of random variables, nonnegative and real ones taking values in \mathbb{R} . Recall that this is the Hamburger case and the M-det criterion is similar to Theorem 13 and omitted. The next result about an M-indet criterion extends slightly Theorem 5.1 of Lin and Stoyanov (2016). The proof is similar and is therefore omitted.

Theorem 15 Consider n independent random variables divided into two groups. The first group, ξ_1, \dots, ξ_{n_0} , consists of nonnegative variables, while all the variables in the second group, $\xi_{n_0+1}, \dots, \xi_n$, take values in \mathbb{R} , where $1 \leq n_0 < n$. Suppose that each $\xi_i \sim F_i$ has a density f_i and that f_i , $i = 1, \dots, n_0$, are positive on $(0, \infty)$, while f_j , $j = n_0 + 1, \dots, n$, are positive on \mathbb{R} and symmetric about 0. Moreover, assume further that

(i) at least one of the densities $f_j(x)$, $j = 1, 2, \dots, n$, is decreasing in $[x_0, \infty)$, where $x_0 \geq 1$ is a constant, and

(ii) for all i , f_i/\bar{F}_i satisfies the condition (5): $f_i(x)/\bar{F}_i(x) \geq A_i/x$ for $x \geq x_0$, and \bar{F}_i satisfies the condition (6): $\bar{F}_i(x) \geq B_i x^{\gamma_i} \exp(-\alpha_i x^{\beta_i})$ for $x \geq x_0$.

If, in addition to the above, $\sum_{i=1}^n 1/\beta_i > 1$, then the product $Z_n = \prod_{i=1}^n \xi_i$ satisfies Krein’s condition, and hence both Z_n and Z_n^2 are M-indet.

An application of the above theorem leads to the following interesting result:

The product of two independent random variables and its square are both M-indet if one random variable is normal and the other is exponential, or chi-square, or inverse Gaussian.

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Competing interests

The author declares that there is no competing interest.

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