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Optimal two-stage pricing strategies from the seller's perspective under the uncertainty of buyer's decisions

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Abstract
In Punta del Este, a resort town in Uruguay, real-estate property is in demand by both domestic and foreign buyers. There are several stages of selling residential units: before, during, and after the actual construction. Different pricing strategies are used at every stage. Our goal in this paper is to derive, under various scenarios of practical relevance, optimal strategies for setting prices within two-stage selling framework, as well as to explore the optimal timing for accomplishing these tasks in order to maximize the overall seller's expected revenue. Specifically, we put forward a general two-period pricing model and explore pricing strategies from the seller's perspective, when the buyer's decisions in the two periods are uncertain: commodity valuations may or may not be independent, may or may not follow the same distribution, be heavily or just lightly influenced by exogenous economic conditions, and so on. Our theoretical findings are illustrated with numerical and graphical examples using appropriately constructed parametric models.

Keywords: Decision theory, Behavioral economics, Uncertainty, Strategy, Two-period economy, Background risk model, Gamma distribution

MSC codes: 62P20, 91A55, 91A80, 91B24

1 Introduction

Commodity pricing has been a prominent topic in the literature, with various models and strategies suggested and explored. In this paper, motivated by a problem described next, we put forward and investigate (both theoretically and numerically) a general model for pricing within the two-period framework that naturally arises in the context of the motivating problem.

1.1 Motivating problem

In Punta del Este, a resort town in Uruguay, real-estate property is in demand by both domestic and foreign buyers. As a recent example, the frequency distribution of buyers for certain high-rise buildings was approximately as follows: 75% Argentineans, 10% Uruguayans, 9% Brazilians, and the remaining ones from the rest of the world (Chile, U.S.A., and so on). A few immediate observations follow. First, the ratio of domestic and foreign buyers varies depending on a number of factors, including economic, financial, and political. Second, it has been observed that the average foreign buyer is wealthier than the average domestic one, and thus tends to exhibit higher bidding prices. Furthermore, given the diversity of buyers, the prices are usually in the US dollars (USD), but some of the building costs such as salaries of workers are in the Uruguayan pesos (UYU).

To properly understand our problem, we need to describe the property development and selling processes. Namely, contracted by an investor, a construction company starts building, say, a residential tower. There are several stages of selling residential units: before, during, and after the actual construction of the tower. Different pricing strategies are used at every stage. It is frequently the case that, at least initially, the investor wishes to sell the units en masse and thus hires a real-estate agent for several months. If the sale is not successful during this initial stage, then the units are put on sale individually, with no particular time horizon set in advance, and at a possibly different price, which could be higher or lower than the original price.

The goal that we set out in this paper is to derive, under various scenarios of practical relevance, optimal strategies for setting first- and second-stage prices, as well as to propose the optimal timing for accomplishing these tasks, in order to maximize the overall seller's expected revenue. In the next subsection, we give a brief appraisal of what we have accomplished in the current paper, with a related though brief literature review given in the following subsection.

1.2 Results and findings – an appraisal

First, in this paper we put forward a highly encompassing, yet tractable, model and explore optimal pricing strategies from the seller's perspective when buyer's real-estate valuations and decisions in the two stages are uncertain: they can be independent or dependent, identically distributed, or stochastically dominate each other, be influenced by exogenous factors at various degrees, and so on. In particular, we shall see from our considerations and examples in the next section that the simultaneous pricing strategies yield higher expected revenues than those under the sequential pricing strategy.

Second, we study the case when real estate costs are possibly denominated in different currencies, as is the case in our motivating problem and, in general, is an important and very common factor in developing countries where large fractions of building costs are denominated in foreign currencies. Hence, currency exchange-rate movements influence optimal pricing decisions.

Third, our model provides conditions under which second-stage prices could be higher or lower than the first-stage prices. This might, initially, be surprising because it is a common intuitive assumption that if a property is not sold during the first stage, then the property price should be reduced before commencing the second stage. As we shall see from our following considerations, however, the relationship between the two stage prices is much more complex: higher holding costs, currency exchange movements, or some type of dominance between the first- and second-stage buyers' bidding price distributions, could very much influence the second-stage price, thus possibly making it larger than that of the first stage, assuming of course that the property was not sold during the first stage.

Finally, our general model accommodates sellers with different shapes of their utility functions, such as those arising in Behavioral Economics (see, e.g., Dhami 2016). In general, while working on this project, we were considerably influenced by, and benefited from, research contributions by many authors, and the following brief literature snapshot highlights some of those that we have found particularly related to the present paper.

1.3 Related literature

House pricing from the seller's and buyer's perspectives has been studied by many authors. For instance, Quan and Quigley (1991), and Biswas and McHardy (2007) adopt the seller's viewpoint in their research. Furthermore, Stigler (1962); Rothschild (1974); Gastwirth (1976); Quan and Quigley (1991); Bruss (2003) and Egozcue et al. (2013) explore the problem from the buyer's perspective. Pricing under different seller's risk attitudes has been studied in the real estate literature as well. For instance, seller's risk neutral behavior has been researched by Arnold (1992; 1999) and Deng et al. (2012). Biswas and McHardy (2007) analyze optimal pricing for risk averse sellers. In addition, Genesove and Mayer (2001); Anenberg (2011) and Bokhari and Geltner (2011) study house price determination for sellers whose risk behavior follows the teachings of Prospect Theory (Kahneman and Tversky 1979).

Bruss (1998, 2003); Egozcue et al. (2013); Egozcue and Fuentes García (2015) and Wu and Zitikis (2017) apply a two-period model to determine optimal commodity (e.g., real estate, computer, etc.) prices that maximize the expected revenue, or minimize the expected loss. Some of the aforementioned works have been influenced by the two-envelope problem, and in particular by the viewpoint put forward by McDonnell and Abbott (2009) and McDonnell et al. (2011). Furthermore, Titman (1985) considers a two-period model to analyze the optimal land prices when the condominium unit prices are uncertain. We also refer to Lazear (1986); Nocke and Peitz (2007); Heidhues and Koszegi (2014), and reference therein, for additional two-period pricing models for real estate.

The rest of this paper is organized as follows. In Section 2 we present several illustrative examples that clarify certain key aspects of our general model proposed in Section 3, such as sequential and simultaneous price settings, differing valuations and thus bid prices, costs associated with holding property unsold. In Section 4 we analyze the first-stage selling probability, and in Section 5 we explore the more complex dynamic second-stage selling probability. In Section 6 we discuss modeling first- and second-stage value functions and then use them to numerically illustrate our general model. Section 7 concludes the paper.

2 Sequential vs simultaneous price setting

In this section we discuss scenarios that clarify various aspects of the problem at hand. In particular, we shall see the difference between setting the two prices sequentially and simultaneously, and we shall also see how the two prices are influenced by considerations such as seeking certain gross or net profits, taking into account possibly different treatments of domestic and foreign buyers, and so on.

We work with a discrete-time two-period economy: $t = 0$ and $t = 1$. Let X_0 and X_1 denote the amounts (i.e., bidding prices) that the buyer is willing to pay for the property during the initial (i.e., $t = 0$) and subsequent (i.e., $t = 1$) selling stages, respectively. Both X_0 and X_1 are random variables from the seller's perspective, and thus we also view them in this way. For the seller, the task is to set an appropriate price p_0 for the initial selling stage, and also an appropriate price p_1 (which is usually different from p_0) for the following selling stage.

It is natural to think that the seller would tend to first set p_0 that would result in a desired outcome such as the maximal expected profit during the initial selling stage, and then, if the sale fails, the seller would set p_1 that would maximize the expected profit during the following selling stage. As we shall illustrate below, the two prices set in this sequential manner may not maximize the expected overall profit, and thus a sensible strategy for the seller who is not in a rush would be to set both p_0 and p_1 before commencing the initial selling stage. The above caveat 'who is not in a rush' is important because rushed decisions usually give rise to very different forces at play, such as willingness to set the price p_0 low enough to ensure a very high probability of selling the property during the initial selling stage. There are of course many other scenarios of practical interest, but in this paper we concentrate on maximizing the expected (gross or net) profit.

The rest of the section consists of two subsections: the first one contains preliminary facts such as sequential and simultaneous pricing, and the second subsection discusses four scenarios that clarify (and justify) the complexity of our general model that we start developing in Section 3.

2.1 Preliminaries

2.1.1 Sequential price setting

Suppose that the seller decides to set the prices p_0 and p_1 sequentially: p_0 before commencing the initial selling stage and p_1 just before the subsequent selling stage. In this case, the maximal expected seller's gross profit during the initial selling stage is the maximal value of the function

$$\Pi_0(p_0) = \mathbf{P}[\,X_0 \geq p_0]\,p_0, \tag{1}$$

which is achieved at the price

$$p_{0,\max} = \arg\max_{p_0} \Pi_0(p_0). \tag{2}$$

Given the sequential manner of setting the prices, the maximal expected seller's gross profit during the second selling stage is the maximal value of the function

$$\Pi_1(p_1) = \mathbf{P}\left[X_0 < p_{0,\max}, X_1 \geq p_1\right] p_1, \tag{3}$$

which is achieved at the price

$$p_{1,\max} = \arg\max_{p_1} \Pi_1(p_1). \tag{4}$$

2.1.2 Simultaneous price setting

The seller may decide to set the two prices p_0 and p_1 simultaneously, before commencing the initial selling stage. In this case, the two expected-profit maximizing prices are

$$\left(p_0^{\max}, p_1^{\max}\right) = \arg\max_{p_0, p_1} \Pi(p_0, p_1), \tag{5}$$

where

$$\Pi(p_0, p_1) = \mathbf{P}\left[X_0 \geq p_0\right] p_0 + \mathbf{P}\left[X_0 < p_0, X_1 \geq p_1\right] p_1. \tag{6}$$

Since $\Pi_0(p_{0,\max}) + \Pi_1(p_{1,\max})$ is equal to $\Pi(p_{0,\max}, p_{1,\max})$, which cannot exceed $\Pi(p_0^{\max}, p_1^{\max})$ by the very definition of $(p_0^{\max}, p_1^{\max})$, the seller cannot be worse off by simultaneously setting the prices before commencing the initial selling stage.

Note 2.1 *The simultaneous setting of prices can be viewed as a strategic decision, whereas setting the prices sequentially just before commencing the respective selling stages are tactical choices, which in view of the above arguments cannot outperform the strategic* *(i.e., simultaneous) one. Deciding on which of these alternatives, and when to make them, has been a prominent topic in the literature, particularly in enterprise risk management (e.g., Fraser and Simkins* 2010*; Segal* 2011*; Louisot and Ketcham* 2014*).*

2.1.3 Gamma distributed bidding prices

To illustrate the above arguments numerically, and to also highlight certain aspects of the general model to be developed later in this paper, in the following subsection we consider four scenarios based on dependent or independent random variables of the form

$$X_0 = a_0 + G_0 \quad \text{and} \quad X_1 = a_0 + G_1,$$

where a_0, which we set to 200 thousands of dollars in our numerical explorations henceforth, is the seller's reservation price during the initial selling stage (i.e., $t = 0$), which is the smallest amount that the seller could possibly ask given the building costs and other expenses, and G_0 and G_1 are two (dependent or independent) gamma distributed random variables.

Although our general model is not limited to any specific price distribution, in our numerical illustrative considerations, we assume that the prices follow the gamma distribution, which is a very reasonable assumption, extensively used in the literature (see, e.g., Pratt et al. 1979; Quan and Quigley 1991; Hong and Shum 2006). In particular, Quan and Quigley (1991) characterize the density function of the reservation price of a group of self-selected buyers using this distribution. Hong and Shum (2006) apply the gamma distribution to model search costs, including time, energy and money spent on researching products, or services, for purchasing. There are numerous cases of using the gamma distribution when modeling insurance losses (e.g., Hürlimann 2001; Furman and Landsman 2005; Alai et al. 2013).

Since different parameterizations of the gamma distribution have appeared in the literature, we note that throughout this paper we work with the one, defined by $Ga(\alpha, \beta)$, whose probability density function (pdf) is[1]

$$f_{\alpha,\beta}(t) = \frac{\beta^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-\beta t}, \quad t > 0. \tag{7}$$

We denote the corresponding cumulative distribution function (cdf) by $F_{\alpha,\beta}$, which for numerical purposes can conveniently be expressed in terms of the lower incomplete gamma function $\gamma(\cdot,\cdot)$ by the formula

$$F_{\alpha,\beta}(x) = \frac{\gamma(\alpha, \beta x)}{\Gamma(\alpha)}. \tag{8}$$

In Fig. 1 we have depicted its pdf for several parameter choices that we use in our numerical explorations later in this paper. The choices are such that we always have the same mean $\mu_G = 50$ of G but varying standard deviations σ_G: equal to 5 (solid), 10 (dashed), 20 (dot-dashed), and 30 (dotted).

2.2 Scenarios

2.2.1 Scenario A: identical X_0 and X_1

Consider the case when the buyer decides on the same bidding price irrespective of the seller's perspective. This bidding price is random, and we denote it by X. In other words, the earlier introduced two random variables X_0 and X_1 are identical, that is, both are equal to a random variable X, which we set to be

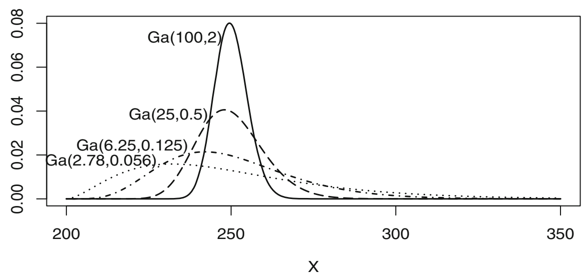

Fig. 1 The pdf of the price $X = 200 + G$ depicted in *solid* $G \sim Ga(100, 2)$, *dashed* $G \sim Ga(25, 0.5)$, *dot-dashed* $G \sim Ga(6.25, 0.125)$, and *dotted* $G \sim Ga(2.78, 0.056)$ lines

$$X = a_0 + G \tag{9}$$

with the earlier defined a_0 and the gamma random variable $G \sim Ga(\alpha, \beta)$. Naturally, if the property is not sold during the initial stage, then under Scenario A, in order to at least hope to be successful during the subsequent stage, the seller has no alternative but to reduce the price, and we shall see this clearly from our following mathematical considerations. We note at the outset, however, that other scenarios to be discussed below will show the possibilities of increasing second-stage prices and still be able to successfully sell the property.

Hence, under Scenario A, and with $p_{0,\max}$ defined by Eq. (2) via the function $\Pi_0(p_0)$ given by Eq. (1) with $X_0 = X$, the function $\Pi_1(p_1)$ is given by the formula

$$\Pi_1(p_1) = \mathbf{P}\left[p_1 \le X < p_{0,\max}\right] p_1. \tag{10}$$

The (simultaneous) expected gross profit $\Pi(p_0, p_1)$, which is defined by Eq. (6) with $X_0 = X_1 = X$, becomes

$$\Pi(p_0, p_1) = \mathbf{P}\left[X \ge p_0\right] p_0 + \mathbf{P}\left[p_1 \le X < p_0\right] p_1. \tag{11}$$

We now use specification (9) to reduce the above formulas to more computationally tractable ones. First, we calculate $p_{0,\max}$, which is the point where the function

$$\Pi_0(p_0) = \left(1 - \frac{\gamma\left(\alpha, \beta(p_0 - a_0)\right)}{\Gamma(\alpha)}\right) p_0 \tag{12}$$

achieves its maximum. Next, we calculate $p_{1,\max}$, which is the point where the function

$$\Pi_1(p_1) = \frac{\gamma\left(\alpha, \beta(p_{0,\max} - a_0)\right) - \gamma\left(\alpha, \beta(p_1 - a_0)\right)}{\Gamma(\alpha)} \mathbf{1}\left\{p_1 \le p_{0,\max}\right\} p_1 \tag{13}$$

achieves its maximum, where the indicator $\mathbf{1}\{p_1 \le p_{0,\max}\}$ is equal to 1 when $p_1 \le p_{0,\max}$ and 0 otherwise. Finally, we calculate the pair $(p_0^{\max}, p_1^{\max})$ that maximizes the function

$$\begin{aligned}\Pi(p_0, p_1) = &\left(1 - \frac{\gamma\left(\alpha, \beta(p_0 - a_0)\right)}{\Gamma(\alpha)}\right) p_0 \\ &+ \frac{\gamma\left(\alpha, \beta(p_0 - a_0)\right) - \gamma\left(\alpha, \beta(p_1 - a_0)\right)}{\Gamma(\alpha)} \mathbf{1}\left\{p_1 \le p_0\right\} p_1.\end{aligned} \tag{14}$$

We report the values of the aforementioned maximal points and the respective expected profits in Table 1. We note that our chosen values of α and β are such that they lead to the same mean $\mu_G = 250$ of G but different standard deviations σ_G $(= \sigma_X)$. We see from the table that we always have $p_{0,\max} > p_{1,\max}$ and $p_0^{\max} > p_1^{\max}$, which is natural because $X_0 = X_1$. As we already mathematically concluded (see below Eq. (6)), the numerical values in

Table 1 Prices and profits when $X_0 = X_1 = X$ with $X = a_0 + G$ and $G \sim Ga(\alpha, \beta)$

α	β	σ_G	$\Pi(p_{0,\max}, p_{1,\max})$	$p_{0,\max}$	$p_{1,\max}$	$\Pi(p_0^{\max}, p_1^{\max})$	$p_0^{\max}$	$p_1^{\max}$
100	2	5	238.3249	238.3686	232.1596	243.6800	246.8699	236.3763
25	0.5	10	229.9880	230.1152	220.5280	238.4024	244.7218	226.8302
6.25	0.125	20	217.1578	217.3732	206.2535	229.8237	241.8907	213.1633
2.78	0.056	30	207.4594	207.5638	200.5773	223.6389	240.9744	204.4544

Table 1 confirm that $p_{0,\max} < p_0^{\max}$ and $p_{1,\max} < p_1^{\max}$, that is, setting the two selling prices simultaneously before commencing the initial selling stage proves to be more beneficial for the seller. Note also from the table that the values of all the four prices $p_{0,\max}$, $p_0^{\max}$, $p_{1,\max}$ and $p_1^{\max}$ decrease when the standard deviation σ_G $(= \sigma_X)$ increases.

When $\alpha = 25$ and $\beta = 0.5$, the functions $\Pi_0(p_0)$ and $\Pi_1(p_1)$ as well as the surface $\Pi(p_0, p_1)$ are depicted in Fig. 2.

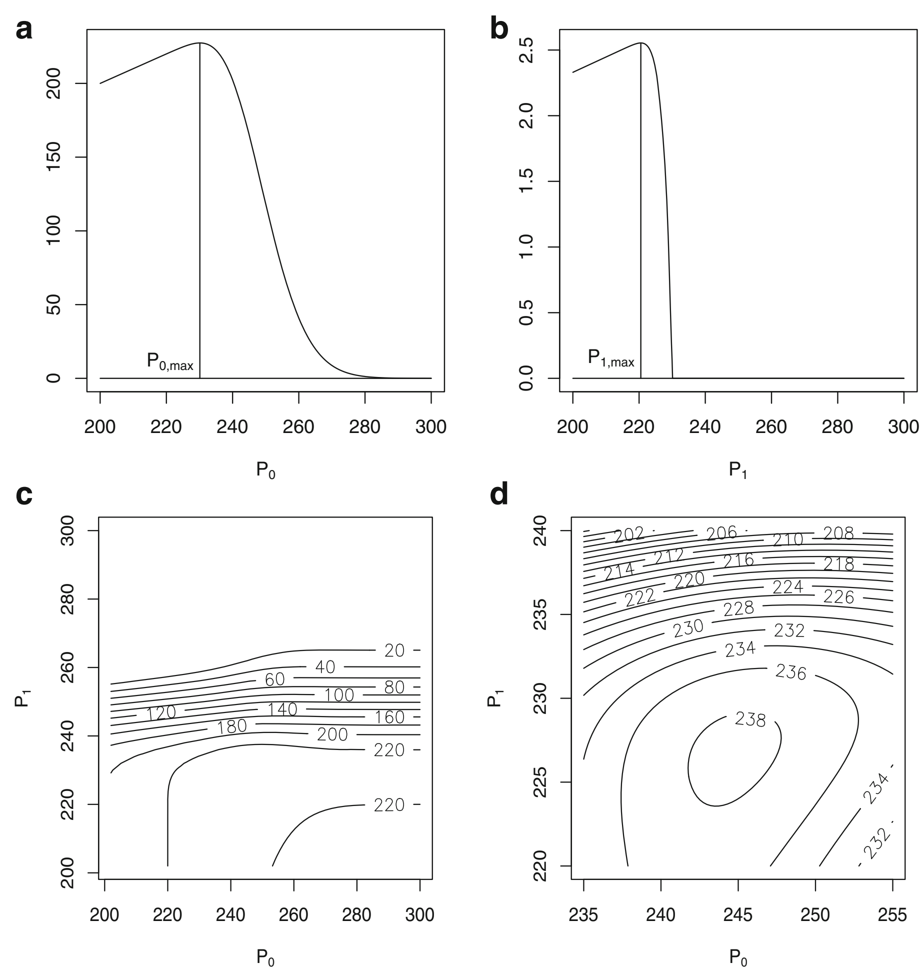

Fig. 2 Prices and gross profits when $X_0 = X_1 = a_0 + G$ with $G \sim Ga(25, 0.5)$. **a** Function $\Pi_0(p_0)$ with $p_{0,\max} = 230.1152$. **b** Function $\Pi_1(p_1)$ with $p_{1,\max} = 220.5280$. **c** Global view of $\Pi(p_0, p_1)$. **d** $\Pi(p_0, p_1)$ around $(p_0^{\max}, p_1^{\max}) = 244.7218, 226.8302$

2.2.2 Scenario B: independent X_0 and X_1

Now we assume that the bidding prices X_0 and X_1 are independent, which sets us apart from Scenario A. However, we still let the two prices follow the same distribution. Specifically,

$$X_0 =_d X \quad \text{and} \quad X_1 =_d X, \tag{15}$$

where $X = a_0 + G$ is the same as in Eq. (9) with $G \sim Ga(\alpha, \beta)$, and '$=_d$' denotes equality in distribution. Hence, $p_{0,\max}$ is defined by Eq. (2) via the function $\Pi_0(p_0)$ given by Eq. (1) with $X_0 = X$, and the expected profits (3) and (6) become

$$\Pi_1(p_1) = \mathbf{P}\left[X < p_{0,\max}\right] \mathbf{P}\left[X \geq p_1\right] p_1 \tag{16}$$

and

$$\Pi\left(p_0, p_1\right) = \mathbf{P}\left[X \geq p_0\right] p_0 + \mathbf{P}\left[X < p_0\right] \mathbf{P}\left[X \geq p_1\right] p_1. \tag{17}$$

Obviously, $p_{0,\max}$ and $p_{1,\max}$ must be identical because X_0 and X_1 follow the same distribution, but there is of course no reason why $p_0^{\max}$ and $p_1^{\max}$ should be identical: the clear difference between the two will be seen from the following numerical example.

First, we see that $p_{0,\max}$ is the same as in Scenario A but $p_{1,\max}$ that maximizes the function

$$\Pi_1(p_1) = \frac{\gamma\left(\alpha, \beta(p_{0,\max} - a_0)\right)}{\Gamma(\alpha)} \left(1 - \frac{\gamma\left(\alpha, \beta(p_1 - a_0)\right)}{\Gamma(\alpha)}\right) p_1 \tag{18}$$

is different from the corresponding one in Scenario A. We see these facts in Table 2 where we use the same shape α and rate β parameters as in earlier Table 1. In Table 2 we have also reported the pairs $(p_0^{\max}, p_1^{\max})$ on which the maximum of the function

$$\begin{aligned}\Pi\left(p_0, p_1\right) = &\left(1 - \frac{\gamma\left(\alpha, \beta(p_0 - a_0)\right)}{\Gamma(\alpha)}\right) p_0 \\ &+ \frac{\gamma\left(\alpha, \beta(p_0 - a_0)\right)}{\Gamma(\alpha)} \left(1 - \frac{\gamma\left(\alpha, \beta(p_1 - a_0)\right)}{\Gamma(\alpha)}\right) p_1\end{aligned} \tag{19}$$

is achieved. Note from Table 1 that the values of all the four selling prices $p_{0,\max}$, $p_0^{\max}$, $p_{1,\max}$ and $p_1^{\max}$ decrease when the standard deviation σ_G $(= \sigma_X)$ increases. Note also that the bounds $p_{0,\max} < p_0^{\max}$ and $p_0^{\max} > p_1^{\max}$ hold. Furthermore, we always see the ordering $p_{0,\max} < p_0^{\max}$ in Table 1, but the ordering of $p_{1,\max}$ and $p_1^{\max}$ seems to depend on the value of σ_G.

In the special case $\alpha = 25$ and $\beta = 0.5$, we have depicted the functions $\Pi_0(p_0)$ and $\Pi_1(p_1)$ as well as the surface $\Pi(p_0, p_1)$ in Fig. 3.

2.2.3 Scenario C: X_1 stochastically dominates X_0

We see from previous two Tables 1 and 2 that neither sequential nor simultaneous second-stage selling prices are higher than the corresponding first-stage prices: we always

Table 2 Prices and gross profits when X_0 and X_1 are independent and follow the distribution of $a_0 + G$ with $G \sim Ga(\alpha, \beta)$

α	β	σ_G	$\Pi(p_{0,\max}, p_{1,\max})$	$p_{0,\max}$	$p_{1,\max}$	$\Pi(p_0^{\max}, p_1^{\max})$	$p_0^{\max}$	$p_1^{\max}$
100	2	5	238.3595	238.3686	238.3686	244.1578	247.0367	238.3651
25	0.5	10	230.0839	230.1152	230.1152	239.2700	244.9522	230.1226
6.25	0.125	20	217.3058	217.3732	217.3732	230.9968	242.0798	217.3781
2.78	0.056	30	207.5201	207.5638	207.5638	224.4556	241.0434	207.5595

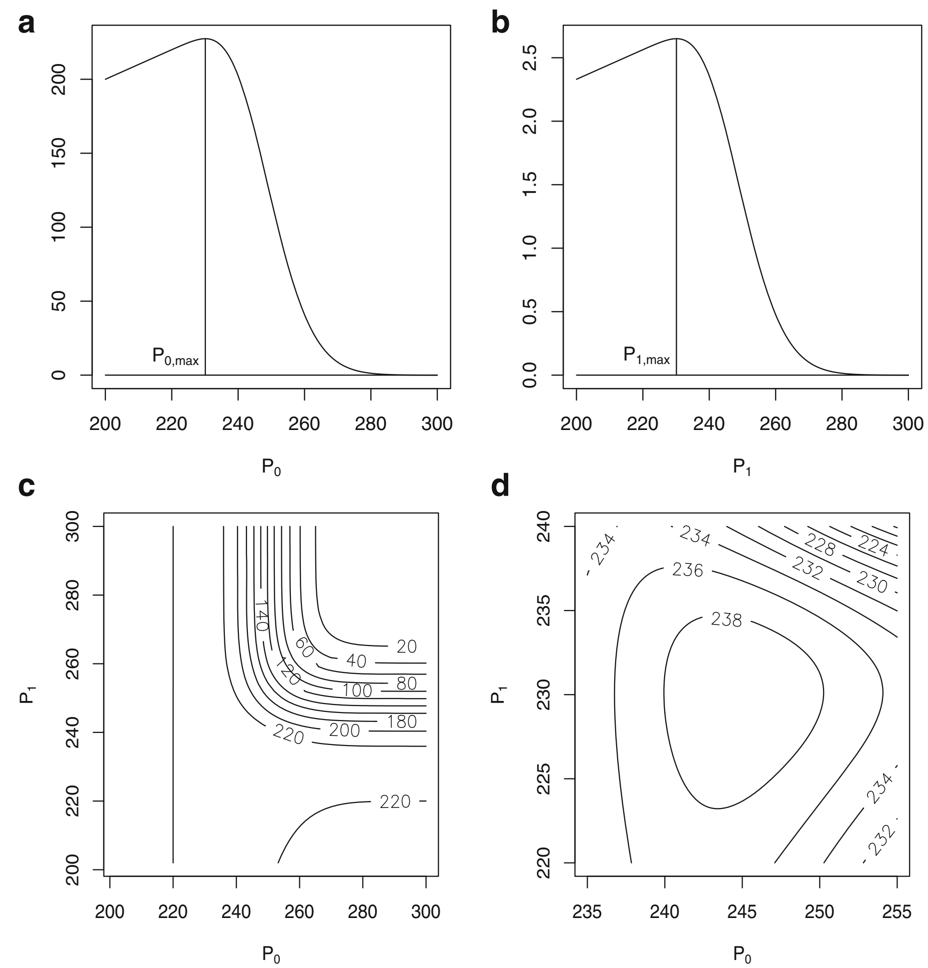

Fig. 3 Prices and gross profits when X_0 and X_1 are independent and follow the distribution of $a_0 + G$ with $G \sim Ga(25, 0.5)$. **a** Function $\Pi_0(p_0)$ with $p_{0,\max} = 230.1152$. **b** Function $\Pi_1(p_1)$ with $p_{1,\max} = 230.1152$. **c** Global view of $\Pi(p_0, p_1)$. **d** $\Pi(p_0, p_1)$ around $(p_0^{\max}, p_1^{\max}) = 244.9522, 230.1226$

have $p_{0,\max} \geq p_{1,\max}$ and $p_0^{\max} \geq p_1^{\max}$ in Tables 1 and 2. In practice, however, we often observe that after the failed initial sales, the sellers increase the prices and achieve successful results. There are several explanations of this phenomenon, and we shall next discuss one of them, with the other one making the contents of Scenario D below.

Namely, our first explanation is based on the assumption that, due to various reasons, buyers are often willing to pay higher prices during the second selling stage. To illustrate this situation numerically, we let

$$X_0 = a_0 + G_0 \quad \text{and} \quad X_1 = a_0 + b_1 G_1,$$

where $b_1 > 0$ is a constant, and $G_0, G_1 \sim Ga(\alpha, \beta)$ are two independent random variables. That is, the buyer is willing to change the bidding amount by $(b_1 - 1)100\%$. Note that $b_1 G_1 \sim Ga(\alpha, \beta/b_1)$, which is useful when calculating. Namely, with the same $p_{0,\max}$ as in Scenarios A and B, we now have

$$\Pi_1(p_1) = \frac{\gamma\left(\alpha, \beta(p_{0,\max} - a_0)\right)}{\Gamma(\alpha)} \left(1 - \frac{\gamma\left(\alpha, \beta(p_1 - a_0)/b_1\right)}{\Gamma(\alpha)}\right) p_1 \tag{20}$$

and

$$\begin{aligned} \Pi\left(p_0, p_1\right) = &\left(1 - \frac{\gamma\left(\alpha, \beta(p_0 - a_0)\right)}{\Gamma(\alpha)}\right) p_0 \\ &+ \frac{\gamma\left(\alpha, \beta(p_0 - a_0)\right)}{\Gamma(\alpha)} \left(1 - \frac{\gamma\left(\alpha, \beta(p_1 - a_0)/b_1\right)}{\Gamma(\alpha)}\right) p_1, \end{aligned} \tag{21}$$

where $p_{1,\max}$ maximizes the function $\Pi_1(p_1)$ and the pair $\left(p_0^{\max}, p_1^{\max}\right)$ maximizes the surface $\Pi(p_0, p_1)$. In Table 3 we have reported the numerical values of the expected gross profits $\Pi(p_{0,\max}, p_{1,\max})$ and $\Pi\left(p_0^{\max}, p_1^{\max}\right)$, as well as of the prices at which these maximal expected gross profits are achieved, for several values of b_1.

We see from Table 3 that for every noted value of b_1, the prices $p_{0,\max}$ and $p_{1,\max}$ decrease when the standard deviation σ_G $(= \sigma_X)$ increases, but the pattern of $p_0^{\max}$ and $p_1^{\max}$ is unclear. Note also from the table that the ordering $p_{0,\max} < p_0^{\max}$ always holds, but various orderings hold between the second-stage prices $p_{1,\max}$ and $p_1^{\max}$. Furthermore, we see that when $b_1 = 0.5$, we have $p_{0,\max} > p_{1,\max}$ and $p_0^{\max} > p_1^{\max}$, but when $b_1 = 1.1$ and $b_1 = 2.1$, we have $p_{0,\max} < p_{1,\max}$ and $p_0^{\max} > p_1^{\max}$.

In the special case $\alpha = 25$, $\beta = 0.5$ and $a = 1.1$, we have depicted the functions $\Pi_0(p_0)$ and $\Pi_1(p_1)$ as well as the surface $\Pi(p_0, p_1)$ in Fig. 4.

2.2.4 *Scenario D: cost of holding the property*

Based on Scenario C, when the seller guesses that the buyer might be willing to pay a large price during the second-stage selling stage, the price in the second stage can be set larger and still the maximal expected gross profit achieved.

There is also another reason why the second-stage selling price can be set larger and the seller's goals achieved, and it is based on the fact that the seller may wish to maximize, for example, the net profit instead of the gross profit. To simplify our illustration of this fact, we take into consideration only one deductible, which is the cost c_1 of holding the property unsold, in which case the (net) profit during the second selling stage becomes $p_1 - c_1$. Furthermore, let the bidding prices X_0 and X_1 be the same as in Scenario B, that is, they are independent and follow the same distribution as $X = a_0 + G$ with $G \sim Ga(\alpha, \beta)$ (see Eq. (15)). Hence, $p_{0,\max}$ is the same as in Scenario B or, equivalently, as in Scenario A,

Table 3 Prices and gross profits when $X_0 = a_0 + G_0$ and $X_1 = a_0 + b_1 G_1$ with independent $G_0, G_1 \sim Ga(\alpha, \beta)$ and varying parameter b_1 values

	α	β	σ_G	$\Pi(p_{0,\max}, p_{1,\max})$	$p_{0,\max}$	$p_{1,\max}$	$\Pi(p_0^{\max}, p_1^{\max})$	$p_0^{\max}$	$p_1^{\max}$
$b_1 = 0.5$	100	2	5	238.2429	238.3686	218.6924	241.1509	243.4476	218.6442
	25	0.5	10	229.9149	230.1152	214.1065	235.7860	240.6705	214.1065
	6.25	0.125	20	217.1670	217.3732	207.1570	228.2490	238.4532	207.1570
	2.78	0.056	30	207.4754	207.5638	202.3953	223.0999	238.8913	202.3953
$b_1 = 1.1$	100	2	5	238.3830	238.3686	242.3547	241.4216	243.7144	220.6309
	25	0.5	10	230.1187	230.1152	233.4259	240.3086	246.1226	233.3253
	6.25	0.125	20	217.3363	217.3732	219.6358	231.6847	242.9417	219.6334
	2.78	0.056	30	207.5313	207.5638	208.8733	224.8072	241.2413	208.8733
$b_1 = 2.1$	100	2	5	238.6217	238.3686	282.7481	241.4216	243.7144	220.6309
	25	0.5	10	230.4780	230.1152	267.7370	250.7337	252.1044	248.5686
	6.25	0.125	20	217.6737	217.3732	245.0225	241.6766	253.5593	244.0345
	2.78	0.056	30	207.6767	207.5638	226.2926	229.9190	247.4248	225.7422

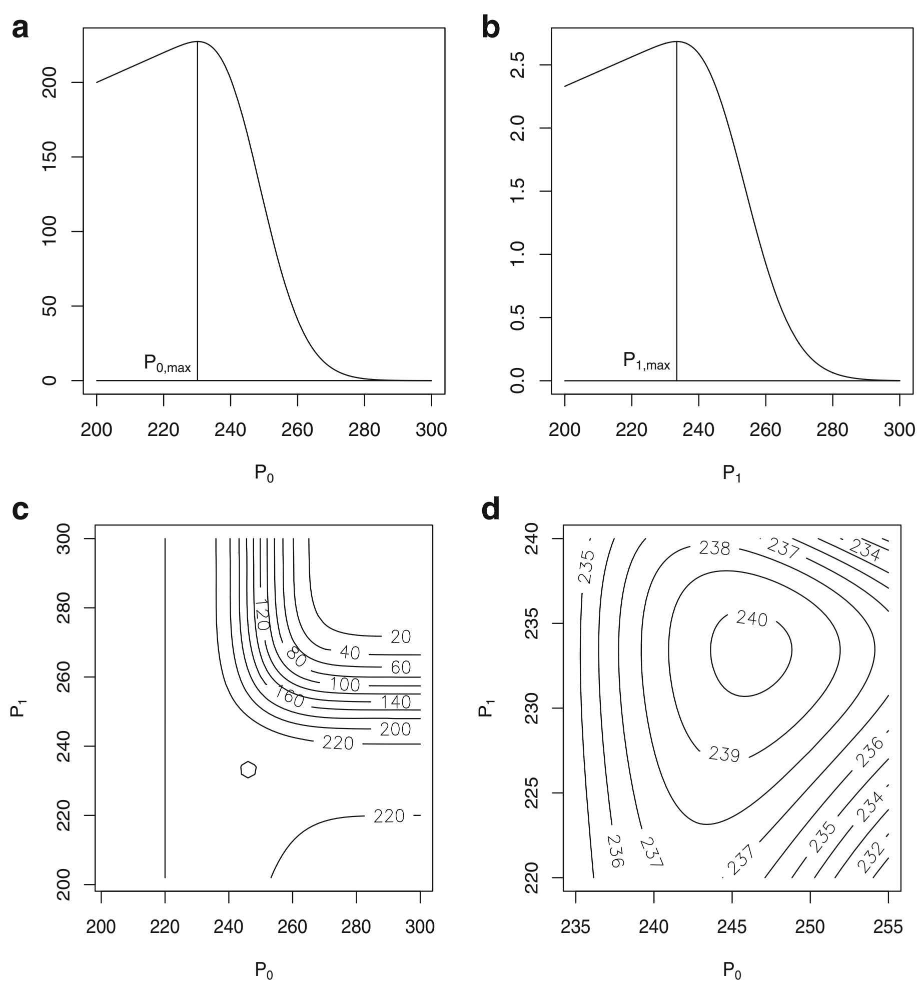

Fig. 4 Prices and gross profits when $X_0 = a_0 + G_0$ and $X_1 = a_0 + b_1 G_1$ with independent $G_0, G_1 \sim Ga(\alpha, \beta)$ and $b_1 = 1.1$. **a** Function $\Pi_0(p_0)$ with $p_{0,\max} = 230.1152$. **b** Function $\Pi_1(p_1)$ with $p_{1,\max} = 233.4259$. **c** Global view of $\Pi(p_0, p_1)$. **d** $\Pi(p_0, p_1)$ around $(p_0^{\max}, p_1^{\max}) = (246.1226, 233.3253)$

that is, the selling price $p_{0,\max}$ is given by Eq. (2) via the same function $\Pi_0(p_0)$ as in Eq. (1). The function $\Pi_1(p_1)$ and the surface $\Pi(p_0, p_1)$, however, need to be redefined in order to take into account the aforementioned cost c_1. Namely, we have

$$\begin{aligned}\Pi_{1,c}(p_1) &= \mathbf{P}\left[X_0 < p_{0,\max}\right] \mathbf{P}\left[X_1 \geq p_1\right](p_1 - c_1) \\ &= \frac{\gamma\left(\alpha, \beta(p_{0,\max} - a_0)\right)}{\Gamma(\alpha)} \left(1 - \frac{\gamma\left(\alpha, \beta(p_1 - a_0)\right)}{\Gamma(\alpha)}\right)(p_1 - c_1),\end{aligned}$$

with the same $p_{0,\max}$ as in Scenario B (or A), and

$$\begin{aligned}\Pi_c(p_0, p_1) &= \mathbf{P}\left[X_0 \geq p_0\right] p_0 + \mathbf{P}\left[X_0 < p_{0,\max}\right] \mathbf{P}\left[X_1 \geq p_1\right](p_1 - c_1) \\ &= \left(1 - \frac{\gamma\left(\alpha, \beta(p_0 - a_0)\right)}{\Gamma(\alpha)}\right) p_0 + \frac{\gamma\left(\alpha, \beta(p_0 - a_0)\right)}{\Gamma(\alpha)} \left(1 - \frac{\gamma\left(\alpha, \beta(p_1 - a_0)\right)}{\Gamma(\alpha)}\right)(p_1 - c_1).\end{aligned}$$

Thus, we have

$$p_{1,c,\max} = \arg\max_{p_1} \Pi_{1,c}(p_1)$$

and

$$\left(p_{0,c}^{\max}, p_{1,c}^{\max}\right) = \arg\max_{p_0,p_1} \Pi_c(p_0,p_1),$$

whose numerical values for different cost c_1 values are reported in Table 4.

We see from the table that for all specified values of c_1, the sequentially set selling prices follow the order $p_{0,\max} < p_{1,c,\max}$, which is the opposite of what we have seen in the previous scenarios. In the case of simultaneously set prices, we have $p_{0,c}^{\max} > p_{1,c}^{\max}$ for the two smaller costs $c_1 = 20$ and $c_1 = 100$, with the opposite ordering $p_{0,c}^{\max} < p_{1,c}^{\max}$ in the case of the cost $c_1 = 150$. The sequentially set selling prices in the initial stage are always smaller than the corresponding simultaneously set prices, that is, the ordering $p_{0,\max} < p_{0,c}^{\max}$ holds throughout the entire table. The reported in Table 4 numerical values of the selling prices $p_{1,c,\max}$ and $p_{1,c}^{\max}$ are very similar.

In the special case $\alpha = 25$, $\beta = 0.5$ and $c_1 = 20$, we have depicted the functions $\Pi_0(p_0)$ and $\Pi_1(p_1)$ as well as the surface $\Pi(p_0,p_1)$ in Fig. 5.

3 The general model

We need to further elaborate on the motivating problem, and to also introduce additional notation. Hence, during the initial selling stage, which we have agreed to collapse into only one instance $t = 0$, the seller keeps the property on sale. Let X_0 be the price, viewed as a random variable, that the buyer is willing to pay for the property during the initial selling stage. Let p_0 be the price set by the seller, who wishes it to be such that certain (economic, financial, etc.) goals would be achieved. Hence, unlike X_0, the price p_0 is not random – the seller chooses it based on the available information and the goals to be achieved. When $X_0 \geq p_0$, the property is sold and the seller's profit is $v_0(p_0)$, where v_0 is a function, usually such that $v_0(p) \leq p$ for all $p \geq 0$. For example,

$$v_0(p_0) = (p_0 - c_0)_+, \tag{22}$$

Table 4 Prices and profits for various holding cost c_1 values when the bidding prices X_0 and X_1 are independent and follow the distribution of $a_0 + G$ with $G \sim Ga(\alpha, \beta)$

	α	β	σ_G	$\Pi(p_{0,\max}, p_{1,c,\max})$	$p_{0,\max}$	$p_{1,c,\max}$	$\Pi(p_{0,c}^{\max}, p_{1,c}^{\max})$	$p_{0,c}^{\max}$	$p_{1,c}^{\max}$
$c_1 = 20$	100	2	5	238.2367	238.3686	238.5186	241.0633	243.3410	238.5186
	25	0.5	10	229.8538	230.1152	230.4159	234.9445	239.6038	230.4159
	6.25	0.125	20	216.9603	217.3732	217.9019	225.1370	234.1515	217.9019
	2.78	0.056	30	207.2347	207.5638	208.1217	217.2595	230.1206	208.1216
$c_1 = 100$	100	2	5	237.7470	238.3686	239.3381	238.2509	239.9507	239.3381
	25	0.5	10	228.9377	230.1152	232.1152	229.8364	233.0915	232.1152
	6.25	0.125	20	215.5916	217.3732	221.1499	216.9273	222.2088	221.1500
	2.78	0.056	30	206.1066	207.5638	212.1154	207.4254	212.7300	212.1154
$c_1 = 150$	100	2	5	237.4428	238.3686	240.2109	237.6034	239.1895	240.2109
	25	0.5	10	228.3730	230.1152	234.0368	228.6428	231.6018	234.0368
	6.25	0.125	20	214.7622	217.3732	225.3874	215.1202	219.6026	225.3874
	2.78	0.056	30	205.4343	207.5638	218.7960	205.7393	209.6705	218.7952

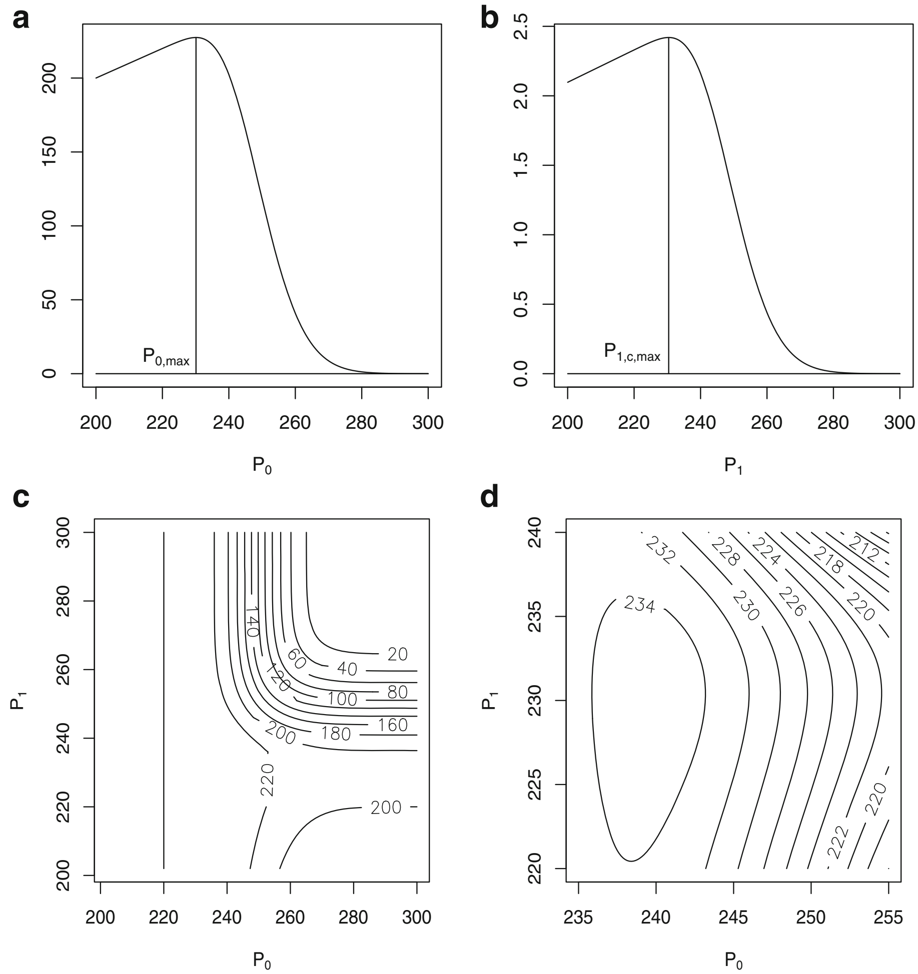

Fig. 5 Profits and prices when the cost is $c_1 = 20$ and the bidding prices X_0 and X_1 are independent and follow the distribution of $a_0 + G$ with $G \sim Ga(\alpha, \beta)$. **a** Function $\Pi_0(p_0)$ with $p_{0,\max} = 230.1152$. **b** Function $\Pi_1(p_1)$ with $p_{1,c,\max} = 230.4159$. **c** Global view of $\Pi(p_0, p_1)$. **d** $\Pi(p_0, p_1)$ around $(p_{0,c}^{\max}, p_{1,c}^{\max}) = (239.6038, 230.4159)$

where c_0 is, e.g., the property development cost evaluated during the initial selling stage. (By definition, $x_+ = x$ when $x \geq 0$, and $x_+ = 0$ when $x < 0$.) If, however, $X_0 < p_0$, then the buyer rejects the offer and makes the second (and final) attempt to buy the property at a later time, which is generally unknown and thus treated as a random variable, which we denote by T.

Note 3.1 *There are of course situations when T is pre-specified and thus deterministic, say $T = 1$. For example, Wu and Zitikis (*2017*) consider a two-period economy with $t = 0$ standing for the Black Friday promotion period and $t = 1$ for the Boxing Day promotion period. In this paper, we let T be random, with specific choices of a distribution and parameter values provided in Note* 6.1 *at the end of this paper.*

Let X_T be the amount of money that the buyer is willing to pay at time $T > 0$ during the second selling stage. Conditionally on T, the price X_T is a random variable from the seller's perspective. Let p_1 be the price set by the seller some time prior to commencing the second selling stage (the price can be set as early as the time of setting the initial price p_0). Analogously to the initial decision making, if $X_T \geq p_1$, then the property is sold and the seller's profit is $v_T(p)$, where v_T is a value (or utility) function, perhaps different from v_0, but usually such that $v_T(p) \leq p$ for all $p \geq 0$. For example, $v_T(p) = (p - c_T)_+$, where c_T is, e.g., the costs of property development and holding it unsold at time T. We shall provide specific details on the structure of c_T later in this paper.

For the sake of concreteness, throughout the rest of the paper we assume that the seller wishes to determine p_0 and p_1 such that the overall two-stage expected profit

$$\begin{aligned}\Pi(p_0,p_1) &= \mathbf{P}\left[X_0 \geq p_0\right] v_0(p_0) + \int_0^\infty \mathbf{P}\left[X_t \geq p_1, X_0 < p_0\right] v_t(p_1)\mathrm{d}F_T(t)\\ &= \mathbf{P}\left[X_0 \geq p_0\right] v_0(p_0) + \left(1 - \mathbf{P}\left[X_0 \geq p_0\right]\right) \int_0^\infty \mathbf{P}\left[X_t \geq p_1 \mid X_0 < p_0\right] v_t(p_1)\mathrm{d}F_T(t)\end{aligned} \tag{23}$$

would be maximal, where F_T is the cdf of T. The seller may have various goals to achieve, and our following considerations can be adjusted accordingly. When deriving Eq. (23), which involves conditioning on T, we have assumed that the events $X_t \geq p_1$ and $T = t$ are independent and in this way obtained the probability $\mathbf{P}[X_t \geq p_1 \mid X_0 < p_0]$.

Even though the simplifying independence assumption is natural, it can be relaxed if a necessity arises, but there are also situations when this assumption is automatically satisfied. For example, this happens in the static two-stage scenario when T always takes the same constant value, say $T = 1$. We note in this regard that the chosen value 1 is just a symbolic representation of the second selling stage, such as the Boxing Day promotion period that follows the initial (i.e., $t = 0$) Black Friday promotion period (e.g., Wu and Zitikis 2017). In this case formula (23) reduces to

$$\Pi(p_0,p_1) = \mathbf{P}\left[X_0 \geq p_0\right] v_0(p_0) + \mathbf{P}\left[X_1 \geq p_1, X_0 < p_0\right] v_1(p_1), \tag{24}$$

where

$$v_1(p_1) = (p_1 - c_1)_+. \tag{25}$$

Henceforth, we shall make a number of other simplifying yet practically sound assumptions, so that the technicalities would not be too complex.

4 The initial-stage selling probability

To assess the probabilities $\mathbf{P}\left[X_0 \geq p_0\right]$ and $\mathbf{P}\left[X_t \geq p_1 \mid X_0 < p_0\right]$ on the right-hand side of Eq. (23), we need to specify appropriate models for the random variables X_t, $t \geq 0$. Their distributions may involve population heterogeneity, as our motivating example shows, which we take into consideration. Specifically, we assume that the population of potential buyers consists of two groups: domestic buyers (D) permanently residing in Uruguay and foreigners (A) wishing to make investments.

Note 4.1 *We have reserved F for denoting cdf's, as is usually the case in the literature, and so use A to denote foreign buyers. This notation also reflects the fact that most of the foreign property buyers in Punta del Este are Argentineans.*

Since economic and financial considerations of the two types of buyers are usually different, the structures of the corresponding random variables are also different. In this section we concentrate on the probability $\mathbf{P}[X_0 \geq p_0]$ and thus specify the structure of X_0. For this, we first note the forces that give rise to the amount of money X_0 that the buyer (domestic or foreign) is willing to pay for the property during the initial selling stage.

In this section and throughout the rest of this paper, background risk models will play an important role. There are two major classes of such models: additive and multiplicative. For applications and discussions of additive models in Economic Theory, we refer to Gollier and Pratt (1996) and references therein, and to problems in Actuarial Science, we refer to Furman and Landsman (2005, 2010); Tsanakas (2008), and references therein. Our current research is essentially based on the multiplicative model, which has been extensively explored and utilized in the literature (see, e.g., Tsetlin and Winkler 2005; Franke et al. 2006, 2011; Asimit et al. 2016; references therein). It is worth noting that a number of important parametric multiplicative models incorporate elements of both Pareto and gamma distributions, and we refer to Asimit et al. (2016); Su (2016) and Su and Furman (2017) for details and further references.

4.1 General considerations

Consider first the population of domestic buyers. Suppose that, initially, their buying decisions are based on individual considerations detached from all the exogenous factors, such as the overall economic situation. Let Y_{0D} be the amount of money (i.e., valuation) that the buyer thinks is affordable and worthy to pay, based on the aforementioned personal considerations. We call Y_{0D} the endogenous domestic valuation.

Naturally, the valuation Y_{0D} is subsequently revised into a more sophisticated and realistic one, which we denote by X_{0D}, taking into account various exogenous factors. We collectively model these factors with a random variable Z_0, that we call the exogenous valuation adjustment. Let h_0 be the function that couples Y_{0D} with Z_0 and gives rise to the aforementioned price X_{0D}, that is,

$$X_{0D} = h_0(Y_{0D}, Z_0). \tag{26}$$

This is the amount of money (i.e., valuation) that the domestic buyer can afford, and is willing, to pay for the property during the initial selling stage.

Likewise, we arrive at

$$X_{0A} = h_0(Y_{0A}, Z_0), \tag{27}$$

which is the amount that the foreign buyer is willing to pay during the initial selling stage, where Y_{0A} is the corresponding endogenous valuation.

Note 4.2 *Throughout this paper we assume that the random variables Y_{0D}, Y_{0A}, and Z_0 are independent, which is a reasonable assumption as we argue next. Indeed, suppose that Y_{0D} and Y_{0A} are dependent. This would suggest that we have not properly separated the exogenous information from the individual valuations of the domestic and foreign buyers, thus contradicting the above description of the endogenous valuations Y_{0D} and Y_{0A}.*

Hence, with X_{0D} representing the amount that the domestic buyer is willing to pay during the initial selling stage, and with X_{0A} representing the corresponding amount of the foreign buyer, the valuation X_0 can be expressed by the formula

$$X_0 = \xi_0 X_{0D} + (1 - \xi_0) X_{0A}, \tag{28}$$

where ξ_0 is a binary random variable taking values 1 and 0, with the event $\xi_0 = 1$ meaning 'domestic buyer.' The proportion of domestic buyers depends on the value of the exogenous valuation adjustment Z_0, which naturally gives rise to the function

$$q_0(z) = \mathbf{P}\left(\xi_0 = 1 \mid Z_0 = z\right),$$

that plays a pivotal role in our subsequent considerations.

Namely, when calculating the probability $\mathbf{P}\left[X_0 \geq p_0\right]$, we first condition on Z_0, whose cdf we denote by F_{Z_0}, and then separate X_{0D} from X_{0A} by conditioning on ξ_0. We obtain the equations

$$\begin{aligned}\mathbf{P}\left[X_0 \geq p_0\right] &= \int \mathbf{P}\left[X_0 \geq p_0 \mid Z_0 = z\right] \mathrm{d}F_{Z_0}(z)\\ &= \int \big(q_0(z)\mathbf{P}\left[X_0 \geq p_0 \mid \xi_0 = 1, Z_0 = z\right]\\ &\quad +(1 - q_0(z))\mathbf{P}\left[X_0 \geq p_0 \mid \xi_0 = 0, Z_0 = z\right]\big)\, \mathrm{d}F_{Z_0}(z). \qquad (29)\end{aligned}$$

Using representation (28) and expressions (26) and (27) on the right-hand side of Eq. (29), we obtain

$$\begin{aligned}\mathbf{P}\left[X_0 \geq p_0\right] &= \int \big(q_0(z)\mathbf{P}\left[X_{0D} \geq p_0 \mid \xi_0 = 1, Z_0 = z\right]\\ &\quad +(1 - q_0(z))\mathbf{P}\left[X_{0A} \geq p_0 \mid \xi_0 = 0, Z_0 = z\right]\big)\, \mathrm{d}F_{Z_0}(z)\\ &= \int \big(q_0(z)\mathbf{P}\left[h_0(Y_{0D}, z) \geq p_0 \mid \xi_0 = 1, Z_0 = z\right]\\ &\quad +(1 - q_0(z))\mathbf{P}\left[h_0(Y_{0A}, z) \geq p_0 \mid \xi_0 = 0, Z_0 = z\right]\big)\, \mathrm{d}F_{Z_0}(z). \qquad (30)\end{aligned}$$

We find it reasonable to simplify the right-hand side of Eq. (30) by first recalling that the endogenous domestic and foreign valuations Y_{0D} and Y_{0A} are independent of the exogenous valuation adjustment Z_0, and then we additionally assume that the valuations Y_{0D} and Y_{0A} do not depend on ξ_0. All of these are justifiable assumptions from the practical point of view. Hence, Eq. (30) simplifies into

$$\mathbf{P}\left[X_0 \geq p_0\right] = \int \big(q_0(z)\mathbf{P}\left[h_0(Y_{0D}, z) \geq p_0\right] + (1 - q_0(z))\mathbf{P}\left[h_0(Y_{0A}, z) \geq p_0\right]\big)\, \mathrm{d}F_{Z_0}(z). \qquad (31)$$

In the next subsection, we specialize formula (31) into a practically sound scenario based on the gamma distribution, under which we subsequently explore the expected profit $\Pi(p_0, p_1)$ numerically and graphically (Section 6 below).

4.2 Specific modelling

The gamma distribution provides a good way to model Y_{0D}, Y_{0A}, and Z_0. In particular, we model the endogenous domestic price Y_{0D} using the shifted gamma distribution supported on the intervals $[\,a_0, \infty)$, with a_0 denoting the seller's reservation price, that is, we have the equation

$$Y_{0D} = a_0 + G_{0D},$$

where $G_{0D} \sim Ga(\alpha_{0D}, \beta_{0D})$. Assuming that the exogenous valuation adjustment Z_0 is an independent gamma random variable $G_0 \sim Ga(\alpha_0, \beta_0)$, the valuation X_{0D} can then be modelled as follows

$$X_{0D} = a_0 + G_{0D}G_0 = h_0(Y_{0D}, Z_0),$$

with the coupling function

$$h_0(y, z) = a_0 + (y - a_0)z. \tag{32}$$

Analogously, starting with

$$Y_{0A} = a_0 + (1 + \varphi_0)G_{0A},$$

where $G_{0A} \sim Ga\,(\alpha_{0A}, \beta_{0A})$ is an independent gamma random variable, with the factor $1 + \varphi_0$ referring to the $(1 + \varphi_0)$ 100% price change (e.g., increase when $\varphi_0 > 0$) that the foreign buyers additionally face when compared to the domestic ones, we arrive at the representation

$$X_{0A} = a_0 + (1 + \varphi_0)G_{0A}G_0 = h_0(Y_{0A}, Z_0),$$

with the same coupling function as in Eq. (32). We have used the same G_0 as in the 'domestic case'.

Note 4.3 *To be in line with our earlier made assumption that foreign buyers generally offer higher endogenous valuations than the domestic ones, in our numerical explorations we choose the gamma parameters so that the average of* $G_{0D} \sim Ga\,(\alpha_{0D}, \beta_{0D})$ *does not exceed the average of* $G_{0A} \sim Ga\,(\alpha_{0A}, \beta_{0A})$*, which is equivalent to bound*

$$\frac{\alpha_{0D}}{\beta_{0D}} \le \frac{\alpha_{0A}}{\beta_{0A}}. \tag{33}$$

Bound (33) *is satisfied for the parameter choices that we shall specify in Note* 6.2 *below.*

Since the random variables G_{0D}, G_{0A}, and G_0 are independent, formula (31) reduces to the following one:

$$\mathbf{P}\left[X_0 \ge p_0\right] = 1 - \int_0^\infty \left\{ q_0(z)F_{\alpha_{0D},\beta_{0D}}\left(\frac{p_0 - a_0}{z}\right) + (1 - q_0(z))F_{\alpha_{0A},\beta_{0A}}\left(\frac{p_0 - a_0}{(1 + \varphi_0)z}\right)\right\} f_{\alpha_0,\beta_0}(z)\mathrm{d}z. \tag{34}$$

It is natural to view the function $q_0(z)$ as decreasing, and such that $q_0(0) = 1$ and $q_0(\infty) = 0$. Thus, for example, we can model $q_0(z)$ as a survival function (i.e., 1 minus a cdf) on the interval $[\,0, \infty)$. The gamma distributions serves a good model, and we thus set

$$q_0(z) = 1 - F_{\gamma_0,\delta_0}(z) \tag{35}$$

in our numerical research later in the paper, with appropriately chosen shape $\gamma_0 > 0$ and rate $\delta_0 > 0$ parameters. For specific parameter choices, we refer to Note 6.2 at the end of this paper.

5 The second-stage selling probability

In this section, we express the probability $\mathbf{P}\left[X_t \ge p_1 \mid X_0 < p_0\right]$ in terms of underlying quantities at every time instance $t > 0$. We accomplish this task in a similar way to that for $\mathbf{P}\left[X_0 \ge p_0\right]$ in the previous section.

5.1 General considerations

We start with additional notations, mimicking the earlier ones. Firstly, we assume that the endogenous valuations Y_{tD} and Y_{tA} as well as the exogenous valuation adjustment Z_t are independent random variables. The definition of the coupling function h_t follows that in Eq. (32) but now with a_t instead of a_0, that is,

$$h_t(y,z) = a_t + (y - a_t)z.$$

Hence, with

$$X_{tD} = h_t(Y_{tD}, Z_t) \quad \text{and} \quad X_{tA} = h_t(Y_{tA}, Z_t),$$

we have

$$X_t = \xi_t X_{tD} + (1 - \xi_t)X_{tA}. \tag{36}$$

Analogously to Eq. (30), we obtain

$$\begin{aligned}\mathbf{P}\left[X_t \geq p_1 \mid X_0 < p_0\right] = \int \Big(& q_t(p_0,z)\mathbf{P}\left[h_t(Y_{tD},z) \geq p_1 \mid X_0 < p_0, \xi_t = 1, Z_t = z\right] \\ &+(1-q_t(p_0,z))\mathbf{P}\left[h_t(Y_{tA},z) \geq p_1 \mid X_0 < p_0, \xi_t = 0, Z_t = z\right]\Big)\, \mathrm{d}F_{Z_t}(z),\end{aligned} \tag{37}$$

where

$$q_t(p_0,z) = \mathbf{P}\left(\xi_t = 1 \mid X_0 < p_0, Z_t = z\right)$$

is the proportion of domestic buyers at time t who did not buy during the initial selling stage (i.e, $X_0 < p_0$).

To make our following considerations simpler, we assume that the endogenous domestic and foreign valuations Y_{tD} and Y_{tA} are based solely on personal considerations at time $t > 0$, that is, they do not depend on any past or current exogenous factors, nor on the past endogenous factors Y_{0D} and Y_{0A}. In other words, we assume that the random variables Y_{tD} and Y_{tA} are independent of X_0, ξ_t and Z_t. This simplifies Eq. (37) into the following one:

$$\begin{aligned}\mathbf{P}\left[X_t \geq p_1 \mid X_0 < p_0\right] = \int \Big(& q_t(p_0,z)\mathbf{P}\left[h_t(Y_{tD},z) \geq p_1\right] \\ &+(1 - q_t(p_0,z))\mathbf{P}\left[h_t(Y_{tA},z) \geq p_1\right]\Big)\, \mathrm{d}F_{Z_t}(z).\end{aligned} \tag{38}$$

5.2 Specific modelling

Analogously to the initial selling stage, we set

$$Y_{tD} = a_t + G_{tD} \quad \text{and} \quad Y_{tA} = a_t + (1 + \varphi_t)G_{tA},$$

where $G_{tD} \sim Ga\left(\alpha_{tD}, \beta_{tD}\right)$ and $G_{tA} \sim Ga\left(\alpha_{tA}, \beta_{tA}\right)$ with the factor $1 + \varphi_t$ referring to the $(1 + \varphi_t)100\%$ additional amount at time t that the foreign buyer needs to pay when compared to the domestic buyer. The exogenous valuation adjustment is

$$Z_t = G_t \sim Ga(\alpha_t, \beta_t).$$

We assume that the three gamma random variables G_{tD}, G_{tA}, and G_t are independent, in which case Eq. (38) reduces to

$$\begin{aligned}\mathbf{P}\left[X_t \geq p_1 \mid X_0 < p_0\right] = 1 - \int_0^\infty &\left\{q_t(p_0,z)F_{\alpha_{tD},\beta_{tD}}\left(\frac{p_1 - a_t}{z}\right)\right.\\ &\left.+(1-q_t(p_0,z))F_{\alpha_{tA},\beta_{tA}}\left(\frac{p_1 - a_t}{(1+\varphi_t)z}\right)\right\} f_{\alpha_t,\beta_t}(z)\mathrm{d}z.\end{aligned} \tag{39}$$

It is reasonable to assume that the seller's reservation price a_t may change over time. For example, it may grow at the inflation rate. Hence, in our numerical explorations we assume that there is a constant ρ such that

$$a_t = (1+\rho t)a_0,$$

for all $t \geq 0$. This assumption reduces Eq. (39) to

$$\begin{aligned}\mathbf{P}\left[X_t \geq p_1 \mid X_0 < p_0\right] = 1 - \int_0^\infty &\left\{q_t(p_0,z)F_{\alpha_{tD},\beta_{tD}}\left(\frac{p_1 - (1+\rho t)a_0}{z}\right)\right.\\ &\left.+(1-q_t(p_0,z))F_{\alpha_{tA},\beta_{tA}}\left(\frac{p_1 - (1+\rho t)a_0}{(1+\varphi_t)z}\right)\right\} f_{\alpha_0,\beta_0}(z)\mathrm{d}z,\end{aligned} \tag{40}$$

where, for the sake of simplicity, we have assumed that the distribution of the exogenous valuation adjustment Z_t does not change with time t, that is, $Z_t \sim Ga(\alpha_0, \beta_0)$ for all $t \geq 0$.

Finally, we introduce an appropriate model for $q_t(p_0, z)$, which is more complex than that for $q_0(z)$. We start with a few observations:

1) When $p_0 = a_0$, it is reasonable to assume that there is not anyone wishing to wait until the second selling stage, and thus $q_t(a_0, z) = 0$ for every exogenous valuation adjustment z.
2) When $p_0 = +\infty$, no one wishes to buy during the initial selling stage, and thus $q_t(+\infty, z)$ should look like $q_0(z)$. Hence, we let $q_t(+\infty, z)$ be the survival function $1 - H_t(z)$ for a cdf $H_t(z)$ on the interval $[0,\infty)$. Just like in the case of $t = 0$, a good model for the cdf H_t is the gamma cdf F_{γ_t,δ_t} with shape $\gamma_t > 0$ and rate $\delta_t > 0$ parameters, which may depend on t.
3) It is reasonable to assume that $q_t(p_0, z)$ is an increasing function of p_0, because larger prices during the initial selling stage would suggest that more domestic buyers are deferring their purchases until the second selling stage.

In summary, we have arrived at the model

$$q_t(p_0, z) = Q_t(p_0 - a_0)(1 - H_t(z)), \tag{41}$$

where Q_t is a non-negatively supported cdf. In Section 6 below, we work with the gamma cdf, that is, we set

$$\begin{aligned}q_t(p_0,z) &= F_{\eta_t,\theta_t}(p_0 - a_0)\left(1 - F_{\gamma_t,\delta_t}(z)\right)\\ &= \frac{\gamma\left(\eta_t, p_0 - a_0\right)}{\Gamma(\eta_t)}\left(1 - \frac{\gamma(\gamma_t,\delta_t z)}{\Gamma(\gamma_t)}\right).\end{aligned} \tag{42}$$

For specific parameter choices, we refer to Note 6.3 at the end of this paper.

6 Value functions and a numerical exploration

To make formula (23) actionable, in addition to the already discussed probabilities $\mathbf{P}\left[X_0 \geq p_0\right]$ and $\mathbf{P}\left[X_t \geq p_1 \mid X_0 < p_0\right]$, we need to specify appropriate models for the value functions $v_0(p_0)$ and $v_t(p_1)$.

6.1 Value function $v_0(p_0)$

We already have a model for $v_0(p_0)$ given by Eq. (22), but in view of our motivating example, an adjustment to this function needs to be made. Namely, property prices in Punta del Este, Uruguay, are predominantly in the US dollars, while property development costs are partially in the Uruguayan pesos and partially in the US dollars. In general, the costs are mainly due to land, design and development, materials, labor costs and subcontracts. Those that are in the Uruguayan pesos are labor costs (i.e., salaries of Uruguayan workers) and they can, for example, be around 30% of the structure's costs, that is, of the total cost minus the land cost. Therefore, we can say that, for some $\nu \in (0, 1)$, the percentage $\nu 100\%$ of the total cost is in the Uruguayan pesos and the rest $(1-\nu)100\%$ is in the US dollars.

To express these costs into one currency, we convert the Uruguayan pesos into the US dollars – because the prices p_0 and p_1 are in the latter currency – using the exchange rate (US dollars per one Uruguayan peso) at an appropriate time instance. Namely, let ε_0 be the exchange rate during the initial selling stage (i.e., $t = 0$). Then Eq. (22) turns into the following one

$$v_0(p_0) = \left(p_0 - \nu c_{0,\mathrm{UYU}}\varepsilon_0 - (1-\nu)c_{0,\mathrm{USD}}\right)_+ . \tag{43}$$

Strictly speaking, the exchange rates are unknown in advance, and thus predicted values need to be used. It is very likely, however, that the prices p_0 and p_1 are set just before commencing the initial selling stage, and thus the value of ε_0 can be reasonably assumed known, and thus v_0 defined in Eq. (43) becomes deterministic and fully specified.

6.2 Value function $v_t(p_1)$

The exchange rate ε_t at time $t > 0$ cannot be known beforehand, that is, at time $t = 0$, and we thus treat it as a random variable. For this reason, we define v_t analogously as v_0, but now with the averaging over the distribution of ε_t, that is, we let

$$\begin{aligned} v_t(p_1) &= \mathbf{E}\left[\left(p_1 - \nu c_{0,\mathrm{UYU}}\varepsilon_t - (1-\nu)c_{0,\mathrm{USD}}\right)_+\right] \\ &= \mathbf{E}\left[\left(p_1 - \nu c_{0,\mathrm{UYU}}\varepsilon_0 r_t - (1-\nu)c_{0,\mathrm{USD}}\right)_+\right], \end{aligned} \tag{44}$$

where $r_t = \varepsilon_t/\varepsilon_0$. In our numerical explorations, we let r_t follows the geometric Brownian motion, that is,

$$r_t = \exp\{\mu t + \sigma W_t\},$$

where W_t is the standard Wiener process (i.e., Brownian motion). This simple model has been a popular example in financial engineering. Equation (44) becomes

$$v_t(p_1) = \mathbf{E}\left[\left(p_1 - \nu c_{0,\mathrm{UYU}}\varepsilon_0 \exp\left\{\mu t + \sigma\sqrt{t}\, N_{0,1}\right\} - (1-\nu)c_{0,\mathrm{USD}}\right)_+\right], \tag{45}$$

where $N_{0,1}$ denotes the standard normal random variable. For specific parameter choices, we refer to Note 6.4 at the end of this paper.

We conclude this section with a note that arguments of Behavioural Economics may suggest using the more general value functions

$$v_0(p_0) = u\left(p_0 - \nu c_{0,\mathrm{UYU}}\varepsilon_0 - (1-\nu)c_{0,\mathrm{USD}}\right)$$

and

$$v_t(p_1) = \mathbf{E}\left[u\left(p_1 - \nu c_{0,\mathrm{UYU}}\varepsilon_0 r_t - (1-\nu)c_{0,\mathrm{USD}}\right)\right]$$

with some function u. Note that we have so far used $u(t) = t_+$, which is a very simple member in the class of so-called S-shaped functions: concave for $t \geq 0$ and convex for $t < 0$. Reverse S-shaped functions, which are convex for $t \geq 0$ and concave for $t < 0$, have also been extensively employed by researchers. We also find many studies where even more complexly shaped functions have been justified. For related discussions, we refer to, for example, Pennings and Smidts (2003); Gillen and Markowitz (2009); Dhami (2016), and references therein.

6.3 A numerical illustration and parameter choices

Using formulas (34), (40), (42), (43) and (45) on the right-hand side of Eq. (23), and with the parameter choices specified below, we obtain an expression for the expected profit $\Pi(p_0, p_1)$ whose maximum with respect to p_0 and p_1 we want to find. Alongside the surface $\Pi(p_0, p_1)$ and the point $(p_0^{\max}, p_1^{\max})$ where it achieves its maximum, in Fig. 6, we have also depicted the profit functions $\Pi_0(p_0)$ and $\Pi_1(p_1)$.

Next are the parameter choices that we have used in our numerical and graphical explorations, summarized in the four panels of Fig. 6 and subsequently detailed in Fig. 7. We note that the parameter choices have arisen from our statistical analyses of (proprietary) data sets, as well as from our Economic Theory based considerations.

Note 6.1 *We assume $T \sim Ga(\alpha_*, \beta_*)$ and set the following parameter values:*

- $\alpha_* = 4$ *and* $\beta_* = 4$

Note 6.2 *These are the specific parameter choices pertaining to the model of Section* 4.2*:*

- $a_0 = 200$
- $\alpha_{0D} = 20$ *and* $\beta_{0D} = 0.6$
- $\alpha_{0A} = 30$ *and* $\beta_{0A} = 0.4$
- $\alpha_0 = \beta_0 = 4$
- $\varphi_0 = 0.2$
- $\gamma_0 = 10$ *and* $\delta_0 = 0.1$

Note 6.3 *These are the specific parameter choices pertaining to the model of Section* 5.2*:*

- $a_t = 200\ (= a_0)$
- $\varphi_t = 0.2$
- $\rho = 0.1$
- $\alpha_{tD} = 20\ (= \alpha_{0D})$ *and* $\beta_{tD} = 0.6\ (= \beta_{0D})$
- $\alpha_{tA} = 30\ (= \alpha_{0A})$ *and* $\beta_{tA} = 0.4\ (= \beta_{0A})$
- $\alpha_t = \beta_t = 4\ (= \alpha_0 = \beta_0)$
- $\eta_t = 8$ *and* $\theta_t = 1$

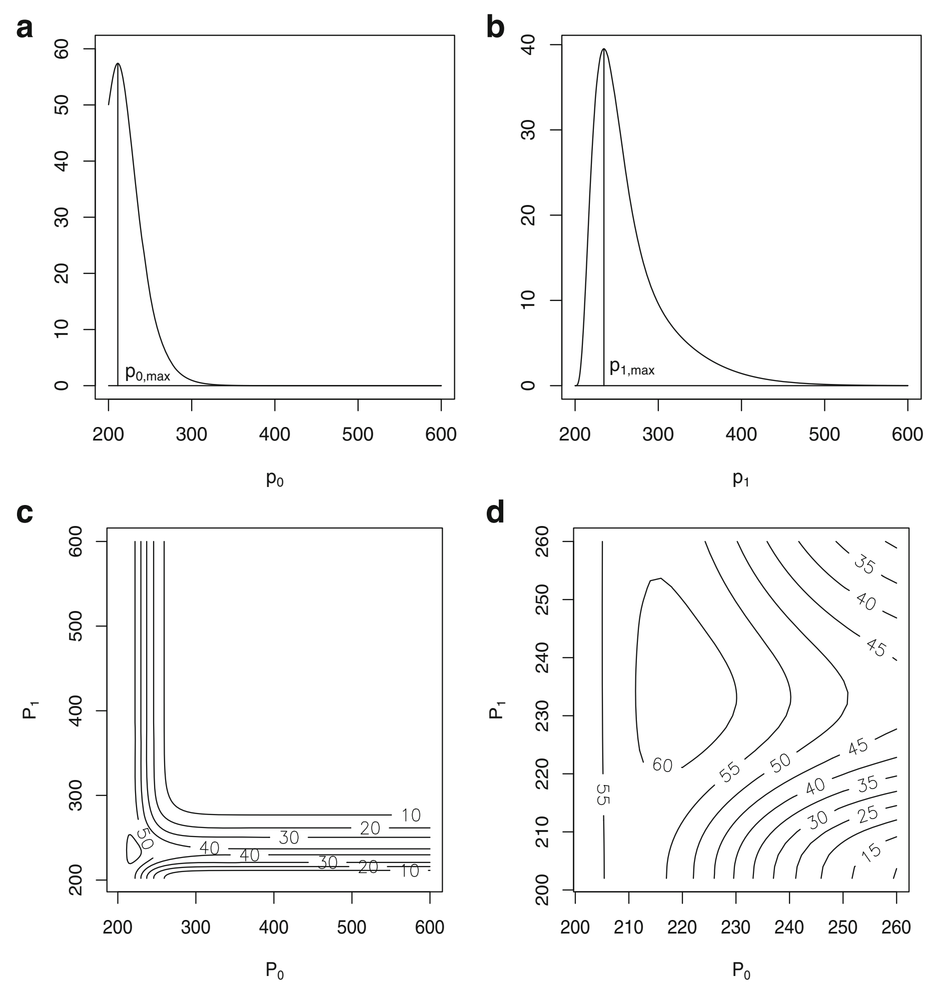

Fig. 6 Profits and prices under the parameter specifications in Notes 6.1–6.4. **a** Function $\Pi_0(p_0)$ with $p_{0,\max} = 211.1910$. **b** Function $\Pi_1(p_1)$ with $p_{1,\max} = 234.6324$. **c** Global view of $\Pi(p_0, p_1)$. **d** $\Pi(p_0, p_1)$ around $(p_0^{\max}, p_1^{\max}) = (219.8946, 233.2321)$

- $\gamma_t = 10$ *and* $\delta_t = 0.1$

Note 6.4 *These are the specific parameter choices pertaining to the value function* $v_t(p_1)$ *discussed in Section* 6*:*

- $\nu = 0.3$
- $c_{0,UYU}\varepsilon_0 = 150$ *and* $c_{0,USD} = 150$
- $\mu = 0$ and $\sigma = 1$

The proposed model has been developed to facilitate well-informed decisions, and the real-life example has guided us in every step of the model development. The model has, inevitably, turned out to be complex. Hence, at this initial stage of our exploration, we have prioritized certain aspects of the research according to their relevance in terms of policy implications, in order to keep considerations within reasonable space limits. The timing of price setting has perhaps been the most significant aspect that is affecting all

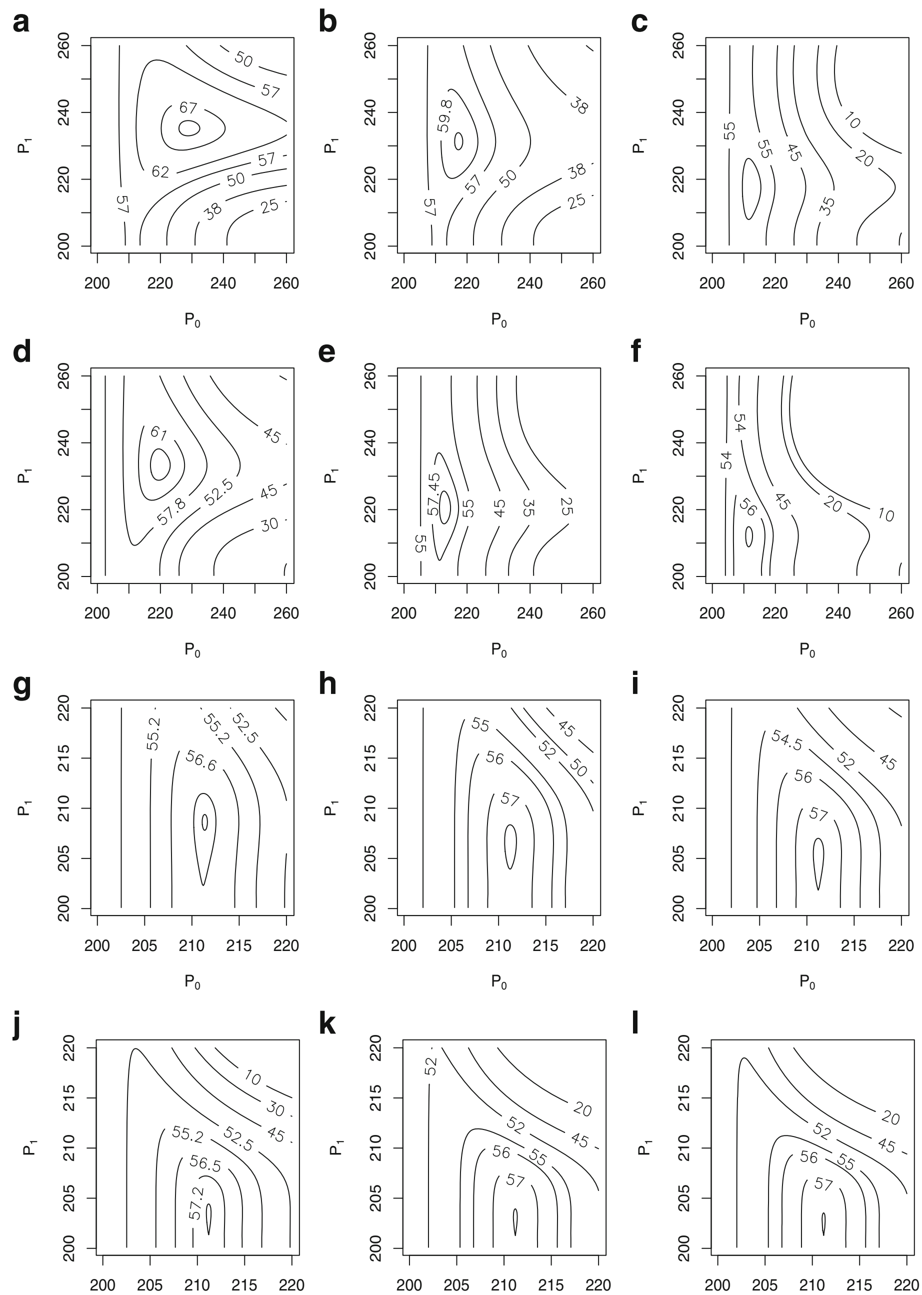

Fig. 7 $\Pi(p_0, p_1)$ for various values of μ and σ. **a** $\mu = 0$ and $\sigma = 0.5$. **b** $\mu = 0.5$ and $\sigma = 0.5$. **c** $\mu = 1$ and $\sigma = 0.5$. **d** $\mu = 0$ and $\sigma = 1$. **e** $\mu = 0.5$ and $\sigma = 1$. **f** $\mu = 1$ and $\sigma = 1$. **g** $\mu = 0$ and $\sigma = 2$. **h** $\mu = 0.5$ and $\sigma = 2$. **i** $\mu = 1$ and $\sigma = 2$. **j** $\mu = 0$ and $\sigma = 3$. **k** $\mu = 0.5$ and $\sigma = 3$. **l** $\mu = 1$ and $\sigma = 3$

the other ones. The dependence between the two-stage pricing decisions and the influence of the systematic (or background) risk has been among the other important aspects. The exchange rate fluctuations, though very important, have nevertheless been given a lesser attention in the present paper, due to a justifiable reason. Namely, a detailed exploration of this aspect with due mathematical care of its various issues such as change points, heteroscedasticity, and other non-linear structures manifesting naturally in financial stochastic models would require considerable space. Our use of the simple geometric Browning motion, instead of a more complex and realistic process, has also been influenced by space considerations. Nevertheless, to give an initial idea about the influence of the mean μ and the volatility σ, we have produced a set of graphs in Fig. 7.

7 Summary

Motivated by a real problem, we have proposed a general two-period pricing model and explored various pricing strategies from the seller's perspective. Our model takes into account such practical considerations as the facts that the buyer's valuations, which are random from the seller's perspective, in the two periods may or may not be independent, may or may not follow the same distribution, and so on. We have seen in particular that the seller's simultaneous-pricing strategies yield higher expected revenues than the sequential-pricing strategies. Our general model allows for the possibility of commodity costs being denominated in different currencies, and thus being impacted by currency exchange-rate movements. The model also takes into account various endogenous and exogenous factors, such as personal seller's and buyer's considerations, general economic conditions, different seller's utility or value functions. We have illustrated our theoretical findings both numerically and graphically, using appropriately constructed multiplicative background models that easily take into account various specific elements of the motivating problem.

Endnote

[1] The mean of this gamma distribution is α/β and the variance is α/β^2.

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Authors' contributions
The authors, ME, JW, and RZ, with the consultation of each other carried out this work and drafted the manuscript together. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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