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On $(p_1, \ldots, p_k)$-spherical distributions

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Abstract

The class of $(p_1, \ldots, p_k)$-spherical probability laws and a method of simulating random vectors following such distributions are introduced using a new stochastic vector representation. A dynamic geometric disintegration method and a corresponding geometric measure representation are used for generalizing the classical χ^2-, t- and F-distributions. Comparing the principles of specialization and marginalization gives rise to an alternative method of dependence modeling.

Keywords: $(p_1, ..., p_k)$-power exponential distribution, simulation of $(p_1, ..., p_k)$-spherical uniform distribution, sign-invariant distribution, $(p_1, ..., p_k)$-spherical surface content measure, $(p_1, ..., p_k)$-spherical radius variable, matrix times vector stochastic representation, dynamic disintegration, $(p_1, ..., p_k)$-spherical coordinates, generalized χ^2-, t- and F-distributions, dependence modeling, specialization vs. marginalization, specialization Copula density, poly Beta function, angular Beta density

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1 Introduction

A basic notion from the theory of spherical probability laws is that of the stochastic basis which is a random vector following the uniform distribution on the Euclidean unit sphere in the k-dimensional Euclidean space $\mathbb{R}^k$, see e.g. Fang et al. (1990). In this monograph, multivariate uniform distributions are introduced in an algebraic way without referring to any type of surface measure. Numerous authors deal with the (singular) uniform distribution by considering the density of its $k-1$-dimensional marginal distribution, an approach which will, however, not further be discussed, here. Instead, the point of view of uniformity of which is the speech here is to define it by having a constant Radon-Nikodym derivative with respect to the Euclidean surface content measure. This geometric view onto the class of spherical distributions is the background of the corresponding geometric measure representation (2) in Richter (1991). This representation extends the one given in (3) in Richter (1985) for the Gaussian law and was exploited later on in a series of papers on probabilities of large deviations and on various statistical distributions. For a related survey see e.g. Richter (2015b).

Many authors studied more general or modified multivariate distribution classes. Just for getting an impression of this research area taking into account different points of view we refer to the more recent papers Field and Genton (2006), Arnold et al. (2008), Kamiya et al. (2008), Balkema et al. (2010), Balkema and Nolde (2010), Richter (2014), Richter (2015a) and Nolan (2016) as well as the references given there.

A detailed geometric description of uniformity of the stochastic basis of $l_{k,q}$-spherical distributions, $q > 0, k \in \{2, 3, \ldots\}$, is given in Richter (2009). The mathematical background there is based upon both a study of the different notions of Euclidean and $l_{k,q}$-surface content measures on $l_{k,q}$-spheres and on using suitable coordinates for evaluating the uniform measure of given subsets of spheres. The coordinates mentioned were introduced in Richter (2007) solving a long standing problem apparently conclusively treated as insolvable in Szablowski (1998).

It is a natural next step of research to consider random vectors having p-spherical uniform distributions with positive components of $p = (p_1, \ldots, p_k)$. The studies of (limit laws in) high risk scenarios in Balkema and Embrechts (2007) and of natural image patches in Sinz et al. (2009), e.g., deliver well motivating examples from this direction. A stochastic matrix times vector and a dynamic geometric measure representation were proved for the corresponding two-dimensional case in Richter (2017). More generally, the necessities of flexible probabilistic modeling in the era of big statistical data make it desirable to further study the class of p-spherical or $l_{k,p}$-symmetric distributions. For technical reasons, we deal exclusively with the case

$$p_i \neq p_j \text{ for } i \neq j, \tag{1}$$

here. As a consequence, distributions studied here are not invariant w.r.t. the class of orthogonal transformations or at least w.r.t. particular rotations but still appear to be sign-invariant meaning invariance w.r.t. multiplication with sign matrixes. While sample schemes for identically distributed variables automatically imply exchangeability of all variables in the present paper this property is excluded due to assumption (1). Distributions considered here are therefore fully outside the scope of the spherical distribution part and in consequence of the remaining parts of the classical monograph by Fang et al. (1990) and numerous work following it.

The present paper is structured as follows. The class of p-spherical uniform distributions is introduced in a geometrically motivated way in Section 2 and extended to the class of p-spherical or $l_{k,p}$-symmetric distributions in Section 3.1. The Sections 3.2 and 3.3 deal with a geometric measure representation and a combination of the principles of specialization and marginalization in dependence modeling, respectively.

2 The class of *p*-spherical uniform distributions

Let us consider the functional

$$|x|^{(p)} = \frac{|x_1|^{p_1}}{p_1} + \ldots + \frac{|x_k|^{p_k}}{p_k}, \; x = (x_1, ..., x_k)^T \in \mathbb{R}^k$$

where the vector $p = (p_1, ..., p_k)$ consists of pairwise different components throughout this paper. The set

$$B_p(r) = \left\{ x \in \mathbb{R}^k : |x|^{(p)} \leq r \right\}$$

will be called the p-ball with p-spherical radius parameter $r > 0$. Clearly, the "unit ball" $B_p = B_p(1)$ is not a norm or antinorm ball, r is not a radius in the sense of Euclidean or any $l_{k,q}$-geometry and the Minkowski functional of $B_p(r)$ is not homogeneous. But if we would allow, for a meantime ignoring assumption (1), to put $p_1 = ... = p_k = q \geq 1$ or $p_1 = ... = p_k = q \in (0, 1)$ then $B_p(r)$ would be a convex norm ball or a radially concave antinorm ball with norm or antinorm radius $(qr)^{1/q}$, respectively. In these cases the notation $B_p(r)$ would coincide with the notation $B_q\left((qr)^{1/q}\right)$ in Richter (2014); Richter (2015a) where q is just a scalar while p is a k-dimensional vector, here. For the notions of antinorm and radial concavity we refer to Moszyńska and Richter (2012). The functional $x \to |x|^{(p)}$ is invariant w.r.t. multiplication with sign matrixes, that is $|Sx|^{(p)} = |x|^{(p)}$ if S is a diagonal matrix with entries $s_1, ..., s_k$ which can be arbitrarily chosen from $\{-1, 1\}$. The topological boundary $S_p(r)$ of $B_p(r)$ is called the p-sphere with p-spherical radius parameter r. We call $S_p(r)$ (positive) matrix-homogeneous meaning that it allows the representation $S_p(r) = D_p(r)S_p$ where

$$D_p(r) = diag\left(r^{\frac{1}{p_1}}, ..., r^{\frac{1}{p_k}}\right)$$

is a diagonal matrix and $S_p = S_p(1)$ denotes the "unit sphere". Note that $B_p(r_1) \subset B_p(r_2)$ if $r_1 < r_2$ and

$$B_p(r) = \bigcup_{\varrho=0}^{r} S_p(\varrho).$$

Let μ and $\mathfrak{B}^k$ denote the Lebesgue measure and the Borel σ-field in $\mathbb{R}^k$, respectively. Assume the random vector X follows the uniform distribution on $\mathfrak{B}(B_p) = \mathfrak{B}^k \cap B_p$, that is

$$P(X \in M) = \frac{\mu(M)}{\mu(B_p)}, M \in \mathfrak{B}(B_p),$$

and let $R^{(p)} = |X|^{(p)}$. What can we say then about the distribution of the random vector

$$U_p = D_p\left(\frac{1}{R^{(p)}}\right)X?$$

The answer to this question is basic for disclosing the main message of this paper and will be given below. The sets

$$CPC_p(A) = \{D_p(r)x : x \in A, r > 0\}$$

and

$$Se_p(A, r) = CPC_p(A) \cap B_p(r)$$

defined for $A \in \mathfrak{B}^k \cap S_p = \mathfrak{B}(S_p)$ are called D_p-transformed central projection cone and D_p-transformed ball sector, respectively. For evaluating the volume of the latter type of sets we shall use the following coordinates.

Definition 1 *Let* $\mathfrak{p} > 0$ *be a parameter and* $M_k = (0,\infty) \times M_k^*$ *where* $M_k^* = [0,\pi)^{\times(k-2)} \times [0, 2\pi)$*. The* $(p,\mathfrak{p})$*-spherical coordinate transformation* $Pol_{p,\mathfrak{p},k} : M_k \to \mathbb{R}^k$ *with*

$$(x_1, ..., x_k)^T = Pol_{p,\mathfrak{p},k}(r, \varphi_1, ..., \varphi_{k-1})$$

is defined by

$$\begin{aligned}
x_1 &= (p_1 r)^{\frac{1}{p_1}} sign(\cos_{\mathfrak{p}} \varphi_1)|\cos_{\mathfrak{p}} \varphi_1|^{\frac{\mathfrak{p}}{p_1}},\\
x_2 &= (p_2 r)^{\frac{1}{p_2}} sign(\cos_{\mathfrak{p}} \varphi_2)(\sin_{\mathfrak{p}} \varphi_1|\cos_{\mathfrak{p}} \varphi_2|)^{\frac{\mathfrak{p}}{p_2}},\\
x_3 &= (p_3 r)^{\frac{1}{p_3}} sign(\cos_{\mathfrak{p}} \varphi_3)(\sin_{\mathfrak{p}} \varphi_1 \sin_{\mathfrak{p}} \varphi_2|\cos_{\mathfrak{p}} \varphi_3|)^{\frac{\mathfrak{p}}{p_3}},\\
&\vdots\\
x_{k-1} &= (p_{k-1} r)^{\frac{1}{p_{k-1}}} sign(\cos_{\mathfrak{p}} \varphi_{k-1})(\sin_{\mathfrak{p}} \varphi_1 \cdot \ldots \cdot \sin_{\mathfrak{p}} \varphi_{k-2}|\cos_{\mathfrak{p}} \varphi_{k-1}|)^{\frac{\mathfrak{p}}{p_{k-1}}},\\
x_k &= (p_k r)^{\frac{1}{p_k}} sign(\sin_{\mathfrak{p}} \varphi_{k-1})(\sin_{\mathfrak{p}} \varphi_1 \cdot \ldots \cdot \sin_{\mathfrak{p}} \varphi_{k-2}|\sin_{\mathfrak{p}} \varphi_{k-1}|)^{\frac{\mathfrak{p}}{p_k}}.
\end{aligned} \tag{2}$$

Here, the q-generalized trigonometric functions $\sin_q \varphi = \frac{\sin \varphi}{N_q(\varphi)}$ and $\cos_q \varphi = \frac{\cos \varphi}{N_q(\varphi)}$ with $N_q(\varphi) = (|\sin \varphi|^q + |\cos \varphi|^q)^{1/q}$, $q > 0$ are introduced in Richter (2007) and used in studying and geometrically representing generalized spherical power exponential distributions in a series of papers starting from Richter (2009).

Remark 1 *The map $Pol_{p,\mathfrak{p},k}$ is almost one-to-one and its inverse is given by*

$$r = \sum_{i=1}^{k} \frac{|x_i|^{p_i}}{p_i}, \tag{3}$$

$$\varphi_i = \arccos_p[\, sign(x_i) \left(\frac{|x_i|^{p_i}/p_i}{\sum_{j=1}^{k} |x_j|^{p_j}/p_j} \right)^{1/\mathfrak{p}}], i = 1, ..., k-2 \tag{4}$$

and

$$\varphi_{k-1} = \arctan \left[sign\left(\frac{x_{k-1}}{x_k}\right) \left(\frac{|x_k|^{p_k}/p_k}{|x_{k-1}|^{p_{k-1}}/p_{k-1}} \right)^{1/\mathfrak{p}} \right]. \tag{5}$$

Evaluating the Jacobian of the transformation $Pol_{p,\mathfrak{p},k}$, the volume of the D_p-transformed ball sector satisfies

$$\mu(Se_p(A, r)) = \frac{1}{\frac{1}{p_1} + ... + \frac{1}{p_k}} r^{\frac{1}{p_1} + ... + \frac{1}{p_k}} \pi_p^*(A) \tag{6}$$

where

$$\pi_p^*(A) = \int\limits_{Pol^{*-1}_{p,\mathfrak{p},k}(A)} J_k^*(\varphi) d\varphi$$

with $J_k^*(\varphi)d\varphi$ being equal to

$$\mathfrak{p}^{k-1} p_1^{\frac{1}{p_1}-1} \cdot ... \cdot p_k^{\frac{1}{p_k}-1} \prod_{i=1}^{k-1} |\cos_{\mathfrak{p}} \varphi_i|^{\frac{\mathfrak{p}}{p_i}-1} |\sin_{\mathfrak{p}} \varphi_i|^{\frac{\mathfrak{p}}{p_k} + ... + \frac{\mathfrak{p}}{p_{i+1}} - 1} \frac{d\varphi_i}{N_{\mathfrak{p}}^2(\varphi_i)}. \tag{7}$$

Here and below, the exponent of $|\sin_{\mathfrak{p}} \varphi_{k-1}|$ is defined to be $\frac{\mathfrak{p}}{p_k} - 1$ and

$$Pol^*_{p,\mathfrak{p},k}(\varphi) = Pol_{p,\mathfrak{p},k}(1, \varphi), \varphi = (\varphi_1, ..., \varphi_{k-1})^T$$

is the restriction of the function $Pol_{p,\mathfrak{p},k}$ to radius parameter 1.

Remark 2 *If the expression in (7) is equivalently rewritten by substituting* $x_i = \cos\varphi_i, i = 1, ..., k$ *then* $\pi_p^*(A)$ *can be represented as*

$$\pi_p^*(A) = \mathfrak{p}^{k-1} p_1^{\frac{1}{p_1}-1} \cdot ... \cdot p_k^{\frac{1}{p_k}-1} \left(\int\limits_{M^+} + \int\limits_{M^-} \right) J_k^{\circledcirc}(x_1, ..., x_{k-1}) d(x_1, ..., x_{k-1}) \tag{8}$$

where

$$J_k^{\circledcirc}(x_1, ..., x_{k-1}) = \prod_{i=1}^{k-1} \frac{|x_i|^{\frac{\mathfrak{p}}{p_i}-1} \left(1 - x_i^2\right)^{\frac{1}{2}\left(\frac{\mathfrak{p}}{p_k}+...+\frac{\mathfrak{p}}{p_{i+1}}\right)-1}}{\left(|x_i|^{\mathfrak{p}} + \left(1 - x_i^2\right)^{\mathfrak{p}/2}\right)^{\frac{1}{p_i}+\frac{1}{p_{i+1}}+...+\frac{1}{p_k}}}$$

and

$$M^{+(-)} = \left\{ (x_1, ..., x_{k-1})^T : \left(x_1, ..., x_{k-1}, +(-) \left(1 - \sum_{j=1}^{k-1} |x_j|^{p_j} \right)^{1/p_k} \right)^T \in A \right\}.$$

The following definition is well motivated by the fact that for the case $p = (q, \dots, q)$ (not considered here) the $l_{k,q}$-surface content measure is similarly introduced in Richter (2009) and proved to be equivalent to the differential-geometric definition. Much more general results of this type of equivalence are proved for norm and antinorm spheres in Richter (2015a).

Definition 2 *Let* $f_A(r) = \mu(Se_p(A, r))$ *for* $r > 0$ *and* $A \in \mathfrak{B}(S_p)$*. We call*

$$O_p(D_p(r)A) = f_A'(r)$$

the p-spherical surface content of $D_p(r)A$ *or its* S_p*-surface content, for short.*

It follows from this definition and equation (6) that

$$O_p(A) = f_A'(1) = \pi_p^*(A).$$

Let us emphasize again that differently from what is assumed in a broad literature p is a vector, here. In particular, as because

$$Pol_{p,\mathfrak{p},k}^{*-1}(S_p) = M_k^*$$

and

$$\int\limits_0^{\pi/2} (\cos_{\mathfrak{p}} \varphi)^{a-1} (\sin_{\mathfrak{p}} \varphi)^{b-1} \frac{d\varphi}{N_{\mathfrak{p}}^2(\varphi)} = \frac{B\left(\frac{a}{\mathfrak{p}}, \frac{b}{\mathfrak{p}}\right)}{\mathfrak{p}} \tag{9}$$

there holds

$$O_p(S_p) = 2^k B\left(\frac{1}{p_1}, ..., \frac{1}{p_k}\right) \prod_{i=1}^{k} p_i^{\frac{1}{p_i}-1}. \tag{10}$$

Here,

$$B(x_1, ..., x_k) = \frac{\Gamma(x_1) \cdot ... \cdot \Gamma(x_k)}{\Gamma(x_1 + ... + x_k)}, x_i > 0, i = 1, ..., k$$

denotes the poly Beta function where $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, x > 0$ is the Gamma function, and $B(.,.)$ is the classical Beta function. Moreover,

$$O_p(D_p(r)A) = \pi_p^*(A) r^{\frac{1}{p_1}+...+\frac{1}{p_k}-1}. \tag{11}$$

Definition 3 *The density*

$$f^*_{\mathfrak{p},a,b}(\varphi) = \frac{\mathfrak{p}}{B\left(\frac{a}{\mathfrak{p}}, \frac{b}{\mathfrak{p}}\right)} \frac{(\cos_{\mathfrak{p}} \varphi)^{a-1} (\sin_{\mathfrak{p}} \varphi)^{b-1}}{N^2_{\mathfrak{p}}(\varphi)}, \varphi \in (0, \pi/2)$$

is called angular Beta density with parameters $\mathfrak{p} > 0$*,* $a > 0$ *and* $b > 0$*.*

Remark 3 *The notion of* S_p*-surface content of* $D_p(r)A$ *is different from the notion of Euclidean surface content of* $D_p(r)A$ *(unless for the case* $p = (2, ..., 2)$ *which, however, because of assumption (1) is not allowed to appear, here).*

Definition 4 *The p-spherical uniform probability law on the Borel* σ*-field* $\mathfrak{B}(S_p)$ *is defined by*

$$\omega_p(A) = \frac{O_p(A)}{O_p(S_p)}.$$

This definition corresponds to Definition 8 in Richter (2007) where, however, p is a scalar while it is a k-dimensional vector, here. Similarly, the notation $U_p = D_p\left(\frac{1}{R^{(p)}}\right)X$ introduced above is closely connected to that given in the same paper where p is a scalar. However, while $R^{(p)} = \frac{|X_1|^{p_1}}{p_1} + ... + \frac{|X_k|^{p_k}}{p_k}$ denotes a certain "mixed-power-of-radius", here, not being the power of a norm or antinorm, in Richter (2007) the symbol R actually means a norm or antinorm. The proof of the following theorem is analogous to that of Theorem 2 in Richter (2017) and will therefore be omitted, here.

Theorem 1 *(a) The random vector* U_p *follows the p-spherical uniform distribution,* $U_p \sim \omega_p$*, is independent of* $R^{(p)}$*, and* $R^{(p)}$ *has the following density with respect to the Lebesgue measure on the real line*

$$\left(\frac{1}{p_1} + ... + \frac{1}{p_k}\right) r^{\frac{1}{p_1}+...+\frac{1}{p_k}-1} I_{[0,1)}(r)dr. \tag{12}$$

(b) If, vice versa, ξ *and* W *are independent where* ξ *has density (12) and* $W \sim \omega_p$ *then* $\eta = D_p(\xi) \cdot W$ *is uniformly distributed on the unit ball* B_p*.*

We are now in a position to disclose the basic message of this paper as follows: matrix-multiplication of a p-spherical uniformly distributed random vector U_p by $D_p(R)$ where U_p and the random variable $R \geq 0$ are independent generates the world of distributions being of main interest, here.

Remark 4 *If the random variable* Y *is uniformly distributed on* $(0, 1)$ *then* $Z_p = Y^{1/\left(\frac{1}{p_1}+...+\frac{1}{p_k}\right)}$ *follows the density (12), that is* $Z_p \overset{d}{=} R^{(p)}$*. Thus* $R^{(p)}$ *can be simulated by* Z_p*.*

Theorem 2 *Let the random vector* X *be uniformly distributed on the unit p-ball* B_p*. Then the p-spherical radius variable* $R^{(p)}$ *and the p-spherical angles* $\Phi_1, ..., \Phi_{k-1}$ *of* X *are* *independent and the angle Φ_i follows the angular Beta density with parameters $\mathfrak{p}$, a_i and b_i, $f_{\mathfrak{p},a_i,b_i}$, where $a_i = \frac{\mathfrak{p}}{p_i}, b_i = \frac{\mathfrak{p}}{p_k} + ... + \frac{\mathfrak{p}}{p_{i+1}}, i = 1, ..., k-2$ and $a_{k-1} = \frac{\mathfrak{p}}{p_{k-1}}, b_{k-1} = \frac{\mathfrak{p}}{p_k}$.*

Proof According to Definition 1, the vector U_p allows the representation

$$U_p = \begin{pmatrix} (p_1)^{\frac{1}{p_1}} \,|\cos_{\mathfrak{p}} \Phi_1|^{\frac{\mathfrak{p}}{p_1}} S_1 \\ (p_2)^{\frac{1}{p_2}} \,|\sin_{\mathfrak{p}} \Phi_1 \cos_{\mathfrak{p}} \Phi_2|^{\frac{\mathfrak{p}}{p_2}} S_2 \\ \vdots \\ (p_k)^{\frac{1}{p_k}} \,|\sin_{\mathfrak{p}} \Phi_1 \sin_{\mathfrak{p}} \Phi_2 \cdot \ldots \cdot \sin_{\mathfrak{p}} \Phi_{k-2} \sin_{\mathfrak{p}} \Phi_{k-1}|^{\frac{\mathfrak{p}}{p_k}} S_k \end{pmatrix} \tag{13}$$

where the signature vector $S = (S_1, \ldots, S_k)^T$ is independent of $\Phi_1, \ldots, \Phi_{k-1}$ and uniformly distributed in $\{-1, 1\}^{\times k}$. It follows from the evaluation of the Jacobian of the transformation $Pol_{p,\mathfrak{p},k}$ that the vector $\left(R^{(p)}, \Phi_1, ..., \Phi_{k-1}\right)^T$ has the density

$$f_p(r, \varphi) = r^{\frac{1}{p_1} + ... + \frac{1}{p_k} - 1} J_k^*(\varphi_1, ..., \varphi_{k-1})$$

where J_k^* satisfies representation (7). For $i = 1, ..., k-1$, the independent angles Φ_i thus have the densities

$$f_{p,\mathfrak{p},i}(\varphi) = \frac{\mathfrak{p}}{B\left(\frac{1}{p_i}, \frac{1}{p_k} + \ldots + \frac{1}{p_{i+1}}\right)} \frac{|\cos_{\mathfrak{p}} \varphi|^{\frac{\mathfrak{p}}{p_i} - 1} |\sin_{\mathfrak{p}} \varphi|^{\frac{\mathfrak{p}}{p_k} + ... + \frac{\mathfrak{p}}{p_{i+1}} - 1}}{N_{\mathfrak{p}}^2(\varphi)}. \tag{14}$$

Now, Definition 3 applies. □

Remark 5 *If Φ_i follows the density in (14) then $Y = |\cos_{\mathfrak{p}} \Phi_i|^{\mathfrak{p}} \sim B(l, m)$ where $l = \frac{1}{p_i}, m = \frac{1}{p_{i+1}} + ... + \frac{1}{p_k}$. Let $(V_{0,1}, V_{0,2})$ be uniformly distributed in $(0,1) \times (0,1)$, then there holds*

$$P(Y \in B) = P\left(\frac{V_{0,1}^{1/l}}{V_{0,1}^{1/l} + V_{0,2}^{1/m}} \in B | V_{0,1}^{1/l} + V_{0,2}^{1/m} \le 1\right), B \in \mathfrak{B} \cap [0, 1].$$

This allows to simulate Y by an acceptance-rejection method, see equation (A5) and algorithm A.1, step 2 in Kalke and Richter (2013). Thus, methods for simulating vectors U_p and X being p-spherical uniformly distributed on S_p and uniformly distributed on B_p, $U_p \sim U(S_p)$ and $X \sim U(B_p)$, respectively, can now be established as follows.

Simulation Algorithm 1[*p*-spherical uniform distribution in $S_p, p = (p_1, ..., p_k)^T$]

Step 1 For $i \in \{1, \ldots, k\text{-}1\}$, simulate $(V_{i,1}, V_{i,2}) \sim U([0, 1] \times [0, 1])$ until $V_{i,1}^{p_i} + V_{i,2}^{1/\left(\frac{1}{p_{i+1}} + ... + \frac{1}{p_k}\right)} \le 1$.

Step 2 Calculate $W_i = \frac{V_{i,1}^{p_i}}{V_{i,1}^{p_i} + V_{i,2}^{1/\left(\frac{1}{p_{i+1}} + ... + \frac{1}{p_k}\right)}}, i = 1, ..., k-1$.

Step 3 Simulate independently $(S_1, \ldots, S_k) \sim U\left(\{-1, +1\}^{\times k}\right)$.

Step 4 Calculate $U_{p,i} = S_i \left(\prod_{j=1}^{i-1} (1 - W_j) W_i\right)^{1/p_i}$ for $i = 1, \ldots, k-1$

and $U_{p,k} = S_k \prod_{j=1}^{k} (1 - W_j)^{1/p_k}$.

Step 5 Return $U_p = (U_{p,1}, ..., U_{p,k})^T$.

Simulation Algorithm 2 [Uniform distribution in $B_p, p = (p_1, ..., p_k)^T$]

Step 1 Simulate $U_p \sim U(S_p)$ according to Algorithm 1.
Step 2 Simulate independently: $Y \sim U(0,1)$.
Calculate $R^{(p)} = Y^{1/(\frac{1}{p_1}+...+\frac{1}{p_k})}$.
Step 3 Return $X = R^{(p)} \cdot U_p$

Remark 6 *By symmetry, the distribution center of* U_p *is* $\mathbb{E}(U_p) = 0_k$ *where* $0_k = (0,...,0)^T \in \mathbb{R}^k$ *is the zero vector of the sample space. The vector* U_p *has uncorrelated components. Using formulas (9), (13) and (14) one can show that the variances of* $U_p's$ *components are*

$$\mathbb{V}(U_{p,i}) = \gamma_1 \cdot \ldots \cdot \gamma_i \cdot p_i^{\frac{2}{p_i}} \frac{\Gamma(\frac{3}{p_i})\Gamma\left(\frac{1}{p_1}+...+\frac{1}{p_k}\right)}{\Gamma(\frac{1}{p_i})\Gamma\left(\frac{3}{p_i}+\sum\limits_{j\neq i}\frac{1}{p_j}\right)}, i=1,...,k$$

where $\gamma_1 = ... = \gamma_{k-2} = 2$ *and* $\gamma_{k-1} = \gamma_k = 4$.

Remark 7 *(a) Let us call*

$$sm_p(A) = \frac{\mu(Se_p(A,1))}{\mu(B_p)}$$

the D_p*-transformed-sector measure on* $\mathfrak{B}(S_p)$ *(or, more precisely, the uniform probability measure of the* D_p*-transformed sector* $Se(A,1)$ *of* B_p*). It follows from the obvious equations*

$$\frac{O_P(A)}{O_P(S_p)} = \frac{f_A'(1)}{f_{S_p}'(1)} = \frac{\pi_p^*(A)}{\pi_p^*(S_p)} = \frac{\mu(Se_p(A,1))}{\mu(Se_p(S_p,1))} = \frac{\mu(Se_p(A,1))}{\mu(B_p)}$$

that

$$\omega_p(A) = sm_p(A).$$

Thus, the p-spherical uniform probability law ω_p *can also be called a* D_p*-transformed-sector measure. For an interpretation of* ω_p *as cone measure see (e) and Remark 9 (b).*

(b) According to Remark 3, the notion of p-spherical uniform distribution on $\mathfrak{B}(S_p)$ *is different from the notion of uniform distribution with respect to Euclidean surface content (unless for* $p_1 = ... = p_k \in \{1, 2, \infty\}$*).*

(c) Fine properties of the Euclidean surface content measure defined on the Borel σ*-field of the Euclidean unit sphere, a precursor of the* S_p*-surface content measure* O_p *considered here, are exploited by the author in the eighties and nineties of the last century in a series of papers on large deviations. A main idea in the background of those work is the development and application of a generalized method of indivisibles extending a classical approach by Cavalieri and Torricelli, see Richter (1985), Richter (2015b) and Günzel et al. (2012).*

(d) It is a challenging problem to find a differential-geometric interpretation of O_p *as it was found in Richter (2009) for the* $l_{k,q}$*-surface content measure and in Richter (2015a) for norm and antinorm spheres. This problem was first stated for the two-dimensional case in Richter (2017).*

(e) *Several authors who study uniform distributions on (non-D_p-transformed) generalized spheres make use of the notion cone measure instead of sector measure and rely on the last representation in (a), that is on relating volumes to each other, see e.g. Naor and Romik (2003) and Barte et al. (2005) and a series of follow up papers.*

(f) *Only a few days before finishing the present paper, Amir Ahmadi-Javid kindly let me know his joint article Ahmadi-Javid and Moeini (2019) where the authors follow another way of considering a uniform distribution on $\mathfrak{B}(S_p)$. They start, in Definition 2.3, with a random vector being uniformly distributed in a parallelepiped and, referring to the work of Schechtman and Zinn (1990), Rachev and Rüschendorf (1991), Song and Gupta (1997), Liang and Ng (2008), Harman and Lacko (2010) and Lacko and Harman (2012), later make use of a common (non-dynamical) notion of cone measure (just like the one mentioned in (e)) for studying a certain type of uniform distributions on $\mathfrak{B}(S_p)$. A closer comparison with the method presented here, where we start with a uniform distribution on a ball B_p and continue with a dynamically transformed-cone measure, would be of interest for future work.*

3 The class of *p*-spherical distributions

A random vector distributed according to the p-spherical uniform distribution builds the stochastic basis of any p-spherical distributed random vector considered in Section 3.1. Examples of light and heavy distribution centers and tails are possible. A geometric measure representation and its applications allow studying exact distributions of generalized χ^2-, t- and F-statistics in Section 3.2. The final Section 3.3 gives a sketch of an alternative approach to describing dependence of random variables following one-dimensional specializations of k-dimensional distributions instead of marginal distributions.

3.1 Definitions and Examples

Definition 5 *Let the random vector U_p follow the p-spherical uniform distribution on the Borel σ-field $\mathfrak{B}(S_p)$, $U_p \sim \omega_p$, and R be a nonnegative random variable having cumulative distribution function (cdf) F and characteristic function (cf) ϕ and being independent of U_p, then*

$$X = D_p(R)U_p \tag{15}$$

is said to follow the p-spherical distribution $\Phi_p^{cdf(F)} = \Phi_p^{cf(\phi)}$. The vector U_p is called the p-spherical uniform basis and R the generating variate of X, and (15) a stochastic representation of X. The distribution of X will alternatively be denoted $\Phi_p^{pdf(f)}$ if R has probability density function (pdf) f.

Remark 8 *If $\mathbb{E}\left(R^{(p)\frac{1}{\min\{p_1,...,p_k\}}}\right)$ is finite then, due to Remark 6, $\mathbb{E}(X) = 0_k$ and if $\mathbb{E}\left(R^{(p)\frac{2}{\min\{p_1,...,p_k\}}}\right) < \infty$ then $\mathbb{V}(X_i) = \mathbb{E}\left(R^{(p)\frac{2}{p_i}}\right)\mathbb{V}(U_{p,i}), i = 1, ..., k$. For the derivation of moments in the case $p_1 = ... = p_k$ we refer to Arellano-Valle and Richter (2012).*

Theorem 3 *The characteristic function of a p-spherically distributed random vector X satisfying representation (15) can be written as*

$$\phi_X(t) = \int_0^\infty \phi_{U_p}(D_p(r)t)P^R(dr),\ t \in R^k$$

where P^R and ϕ_{U_p} denote the distribution law of R and the characteristic function of U_p, respectively.

Proof Because of the diagonal structure of $D_p(r)$ we have

$$\phi_X(t) = \mathbb{E}e^{i(t,D_p(R)U_p)} = \mathbb{E}e^{i(D_p(R)t,U_p)}.$$

If $\mathbb{E}(Y|R)$ denotes the conditional expectation of Y given $R = r$ then

$$\phi_X(t) = \mathbb{E}\mathbb{E}\left(e^{i(D_p(r)t,U_p)}|R = r\right) = \int_0^\infty \phi_{U_p}(D_p(r)t)P^{R|U_p}(dr)$$

from where the result follows by independence of U_p and R. □

Corollary 1 *(a) The distribution of a p-spherically distributed random vector X is uniquely determined by the distribution of its generating variate R.*

(b) If a p-spherically distributed random vector X has a density, then it is of the form $f_X = \varphi_{g;p}$,

$$\varphi_{g;p}(x) = C(g;p)g\left(|x|^{(p)}\right), x \in \mathbb{R}^n$$

where $g : [0,\infty) \to [0,\infty)$ is a density generating function (dgf) satisfying

$$0 < I(g;p) = \int_0^\infty r^{\frac{1}{p_1}+\ldots+\frac{1}{p_k}-1}g(r)dr < \infty,$$

and the normalizing constant allows the factorization

$$\frac{1}{C(g;p)} = I(g;p)O_p(S_p).$$

This density is invariant w.r.t. multiplication with sign matrices, or sign-invariant or sign-symmetric. For a general class of symmetric distributions we refer to Arellano-Valle et al. (2002) and Arellano-Valle and del Pino (2004).

Both this result and the following definition transfer earlier statements from Richter (2014) to the present case. The following definition adopts notation in Müller and Richter (2016) and is aimed to make the notion of dgf unique.

Definition 6 *A dgf g satisfying the equation*

$$I(g;p)O_p(S_p) = 1 \quad (16)$$

is called density generator (dg) of a continuous p-spherical distribution.

Example 1 *(a) The dg of the p-spherical Kotz type distribution having parameters $M > 1 - \frac{1}{p_1} - \ldots - \frac{1}{p_k}$ and β and γ from $(0,\infty)$ is*

$$g^{(p)}_{Kt;M,\beta,\gamma}(r) = C^{(p)}_{Kt;M,\beta,\gamma} r^{M-1}e^{-\beta r^\gamma} I_{(0,\infty)}(r)$$

where

$$C^{(p)}_{Kt;M,\beta,\gamma} = \frac{\gamma\beta^{\left(M-1+\frac{1}{p_1}+...+\frac{1}{p_k}\right)/\gamma}}{\Gamma\left(\left(M-1+\frac{1}{p_1}+...+\frac{1}{p_k}\right)/\gamma\right)2^k B\left(\frac{1}{p_1},...,\frac{1}{p_k}\right)\prod_{i=1}^{k} p_i^{1/p_i-1}}.$$

The p-spherical Kotz type pdf with parameters M, β, γ *is therefore*

$$\varphi^{(p)}_{Kt;M,\beta,\gamma}(x) = C^{(p)}_{Kt;M,\beta,\gamma}\left(\frac{|x_1|^{p_1}}{p_1}+...+\frac{|x_k|^{p_k}}{p_k}\right)^{M-1} e^{-\beta\left(\frac{|x_1|^{p_1}}{p_1}+...+\frac{|x_k|^{p_k}}{p_k}\right)^{\gamma}}. \tag{17}$$

(b) Particularly, the dg of the p-spherical power exponential distribution is

$$g^{(p)}_{PE}(r) = \left(\prod_{i=1}^{k} C_i\right) e^{-r} I_{(0,\infty)}(r)$$

and one may write then

$$\Phi^{(p)}(dx) = \left(\prod_{i=1}^{k} C_i\right)\exp\left\{-|x|^{(p)}\right\} dx_1...dx_k$$

to denote the k-dimensional p-power exponential density where $C_i = \frac{p_i^{1-1/p_i}}{2\Gamma(1/p_i))}$, $i = 1,...,k$ *are individual normalizing constants. In this case the random vector X consists of independent components, and according to (a) there holds*

$$\prod_{i=1}^{k} C_i = C^{(p)}_{Kt;1,1,1}.$$

Example 2 *The dg of the p-spherical Pearson Type VII distribution having parameters* $M > \max\left\{1, \frac{1}{p_1}+...+\frac{1}{p_k}\right\}$ *and* $\nu > 0$ *is*

$$g^{(p)}_{PT7;M,\nu}(r) = C^{(p)}_{PT7;M,\nu}\left(1+\frac{r}{\nu}\right)^{-M} I_{(0,\infty)}(r)$$

where

$$C^{(p)}_{PT7;M,\nu} = \frac{\Gamma(M)\prod_{i=1}^{k} p_i^{1-1/p_i}}{2^k \nu^{\frac{1}{p_1}+...+\frac{1}{p_k}}\cdot\Gamma\left(M-\frac{1}{p_1}-...-\frac{1}{p_k}\right)\Gamma\left(\frac{1}{p_1}\right)\cdot\dots\cdot\Gamma\left(\frac{1}{p_k}\right)}.$$

The p-spherical Pearson Type VII density with parameters M and ν *is therefore*

$$\varphi^{(p)}_{PT7;M,\nu}(x) = C^{(p)}_{PT7;M,\nu}\left(1+\frac{1}{\nu}\left(\frac{|x_1|^{p_1}}{p_1}+...+\frac{|x_k|^{p_k}}{p_k}\right)\right)^{-M}. \tag{18}$$

3.2 Geometric measure representation

The following theorem is a geometric-measure theoretic counterpart to Theorem 3. Its proof follows the line of author's earlier work in Richter(1985, 1991, 2014, 2017). The main aim of this section is to present first applications of the geometric measure representation extending the classical Helmert-Pearson χ^2-, Gosset alias Student t- and Fisher F-distributions.

Let $\Phi^{(p)}_g$ denote the continuous p-spherical distribution law having dg g.

Theorem 4 *For every $B \in \mathfrak{B}^k$,*

$$\Phi_g^{(p)}(B) = \frac{1}{I(g;p)} \int_0^\infty r^{\frac{1}{p_1}+...+\frac{1}{p_k}-1} g(r) \mathfrak{F}_p(B,r) dr.$$

Let a random vector X follow the p-spherical distribution law with dg g, $X \sim \Phi_{g;p}$, and $T = T(X)$ be any statistic. The statistic T satisfies $T < \lambda$ if and only if the outcome of X belongs to the sub-level set

$$B_T(\lambda) = \left\{ x \in \mathbb{R}^k : T(x) < \lambda \right\},$$

thus

$$P(T < t) = \frac{1}{I(g;p)} \int_0^\infty r^{\frac{1}{p_1}+...+\frac{1}{p_k}-1} g(r) \mathfrak{F}_p(B_T(\lambda), r) dr. \tag{19}$$

Example 3 ***Chi-p statistic*** *Let*

$$T(X) = |X|^{(p)}$$

denote the p-spherical radius variable of the random vector X, that is $T(X) = R^{(p)}$, then the ipf of the set B_T satisfies

$$\mathfrak{F}_p(B_T(\lambda), r) = I_{(0,\lambda)}(r), r > 0.$$

The Chi-$(g;p)$ pdf is therefore

$$f(\lambda) = \lambda^{\frac{1}{p_1}+...+\frac{1}{p_k}-1} g(\lambda) I_{(0,\infty)}(\lambda). \tag{20}$$

Differently from the $\chi^2(k)$-distribution where the parameter k corresponds to the dimension of a subspace of the sample space, here, the parameter $\frac{1}{p_1}+...+\frac{1}{p_k}$ itself does not allow interpretation as dimension of a linear space or a linear subspace of the sample space, but its number of summands k does.

Note that the density of T was dealt with for the particular dg of the generalized Gaussian law in Taguchi (1978). For $\frac{1}{p_1}+...+\frac{1}{p_k} = \frac{k}{2}$ the pdf in (20) is called the g-generalization of the Chi-square density with k degrees of freedom (d.f.) in Richter (1991) (note that there holds $p_1 = \ldots = p_k$ but (16) is not assumed to be satisfied there). For more partial cases and statistical applications of this distribution, see Richter (2007, 2009, 2016).

If we specify $g = g_{PE}^{(p)}$ in (19), see Example 1(b), then

$$f_{R^{(p)}}(\lambda) = \lambda^{\frac{1}{p_1}+...+\frac{1}{p_k}-1} e^{-\lambda} \prod_{i=1}^k \frac{p_i^{1-1/p_i}}{2\Gamma(1/p_i)}.$$

In this case we have

$$\mathbb{E}R^{(p)} = \Gamma\left(\frac{1}{p_1}+...+\frac{1}{p_k}+1\right) \prod_{i=1}^k \frac{p_i^{1-1/p_i}}{2\Gamma(1/p_i)}$$

and

$$\mathbb{E}(R^{(p)2}) = \Gamma\left(\frac{1}{p_1}+...+\frac{1}{p_k}+2\right) \prod_{i=1}^k \frac{p_i^{1-1/p_i}}{2\Gamma(1/p_i)}$$

as well as

$$\mathbb{E}(R^{(p)\frac{2}{p_i}}) = \Gamma\left(\frac{3}{p_i} + \sum_{j\neq i}\frac{1}{p_j}\right)\prod_{j=1}^{k}\frac{p_j^{1-1/p_j}}{2\Gamma(1/p_j)}.$$

It follows from Remarks 6 and 8 that

$$\mathbb{V}(X_i) = \frac{\gamma_1\cdot\ldots\cdot\gamma_i}{2^k B\left(\frac{1}{p_1},\ldots,\frac{1}{p_k}\right)}\frac{\Gamma\left(\frac{3}{p_i}\right)}{\Gamma\left(\frac{1}{p_i}\right)}p_i^{1+\frac{1}{p_i}}\prod_{j\neq i}p_j^{1-\frac{1}{p_j}}. \tag{21}$$

Similarly, if we put $g = g^{(p)}_{Kt;M,\beta,\gamma}$ *or* $g = g^{(p)}_{PT7;M,\nu}$ *in (19) then* $f_{R^{(p)}}$ *and* $\mathbb{V}(X_i)$ *will be correspondingly specified.*

Example 4 ***Fisher-p statistic*** *Let the vectors* $X^{(1)T} = (X_1,\ldots,X_m)$ *and* $X^{(2)T} = (X_{m+1},\ldots,X_k)$ *be sub-vectors of* $X = (X_1,\ldots,X_m,X_{m+1},\ldots,X_k)^T$ *and* $p^{(1)} = (p_1,\ldots,p_m)^T$, $p^{(2)} = (p_{m+1},\ldots,p_k)^T$ *be sub-vectors of the shape-tail parameter vector* $p = (p_1,\ldots,p_m,p_{m+1},\ldots,p_k)^T$, *and assume that* $X\sim\Phi_{g;p}$. *We consider the F-p statistic*

$$T(X) = \frac{|X^{(1)}|^{(p_1)}/m}{|X^{(2)}|^{(p_2)}/(k-m)}$$

and recognize that $T(x) = T(D_p(\gamma)x)$ *for all* $\gamma > 0$. *Roughly spoken,* B_T *has the curved cone-type property*

$$D_p(\gamma)B_T(\lambda) = B_T(\lambda), \lambda > 0. \tag{22}$$

Thus the ipf of the set B_T *does not depend on r and equation (19) shows that*

$$P(T<\lambda) = \mathfrak{F}_p(B_T(\lambda),1).$$

Making use of the coordinate transformations

$$Pol_{p^{(1)},\mathrm{p},m} : (r_1,\varphi_1,\ldots,\varphi_{m-1}) \to x^{(1)}$$

and

$$Pol_{p^{(2)},\mathrm{p},m} : (r_2 = \cos_1\varphi_{m+1},\ldots,\varphi_{k-1}) \to x^{(2)}$$

instead of the coordinate transformations $SPH_{p,1}$ *and* $SPH_{p,2}$ *used in Richter (2009), and*

$$r_1 = \cos_1\varphi, r_2 = \sin_1\varphi, 0\le\varphi<\pi/2,$$

and further following the line of the proof of Theorem 6 there, we get

$$\mathfrak{F}_p(B_T(\lambda),1) = \frac{1}{B\left(\frac{1}{p_1}+\ldots+\frac{1}{p_m},\frac{1}{p_{m+1}}+\ldots+\frac{1}{p_k}\right)} \cdot \int_{arc\cot\left(\frac{m\lambda}{k-m}\right)}^{\pi/2}\frac{(\cos\varphi)^{\frac{1}{p_1}+\ldots+\frac{1}{p_m}-1}(\sin\varphi)^{\frac{1}{p_{k+1}}+\ldots+\frac{1}{p_k}-1}}{N_1(\varphi)^{\frac{1}{p_1}+\ldots+\frac{1}{p_k}}}d\varphi.$$

Taking the derivative shows that the pdf of statistic T is

$$f_{m,k-m}(\lambda) = \frac{\left(\frac{m}{k-m}\right)^{\frac{1}{p_1}+\ldots+\frac{1}{p_m}}\lambda^{\frac{1}{p_1}+\ldots+\frac{1}{p_m}-1}}{B\left(\frac{1}{p_1}+\ldots+\frac{1}{p_m},\frac{1}{p_{m+1}}+\ldots+\frac{1}{p_k}\right)\left(1+\left(\frac{m\lambda}{k-m}\right)\right)^{\left(\frac{1}{p_1}+\ldots+\frac{1}{p_k}\right)}}. \tag{23}$$

We call the pdf in (23) the Fisher-p density with $(m, k-m)$ *d.f. or* $F_{m,k-m}(p)$*-density, for short.*

Example 5 ***Student-p statistic*** *Let the t-p statistic be defined by*

$$T(X) = \frac{X_1}{\left(|X^{(2)}|^{(p_2)}/(k-1)\right)^{1/p_1}}.$$

The pdf of T is called Student p-density with $k-1$ *d.f. or* $t_{k-1}\left(\frac{1}{p_1},\ldots,\frac{1}{p_k}\right)$*-density, for short:*

$$\frac{d}{dt}P(T<t) = \frac{p_1}{2}f_{1,k-1}(t^{p_1})t$$
$$= \frac{p_1}{2(k-1)^{\frac{1}{p_1}}B\left(\frac{1}{p_1},\frac{1}{p_2}+\ldots+\frac{1}{p_k}\right)(1+\frac{|t|^{p_1}}{k-1})}.$$

Note that, as in Example 4, B_T *is a* D_p*-transformed-cone or curved-transformed-cone type set satisfying (22).*

Remark 9 *(a) Because Fisher-p and Student-p distributions do not depend on the dg, the F-p and t-p statistics are called g-robust. For a study of g-sensitivity and g-robustness of certain statistics see Ittrich et al. (2000) and for a study of statistics generating curved-transformed-cone type sets, see Ittrich and Richter (2005).*

(b) Let $B_T(\lambda)$ *be a curved-transformed-cone type set satisfying (22) and put* $A = B_T(\lambda)\cap S_p$*. By Theorem 4,* $\Phi_g^{(p)}(B_T(\lambda)) = \omega_p(A)$*. It is reasonable therefore to call* $C^{(p)}(A) = \omega_p(A)$ *the* D_p*-transformed-cone measure of* $A \in \mathfrak{B}(S_p)$*.*

(c) We recall that several representations of Student distributed statistics were given in Richter (1995). In particular, the two facts are exploited there that the ipf of the cone $\{T<\lambda\}$ *does not depend on its radius variable and that the multivariate standard Gaussian law is invariant w.r.t. orthogonal transformations, together leading to g-robustness of Fisher's and Student's statistics. Due to assumption (22), we observe in the present situation that the ipf of* $B_T(t)$ *does not depend on the generalized radius variable and we observe invariance of Fisher's and Student's statistics w.r.t. any transformation* $D_p(r)$ *where* $r>0$ *is used to prove g-robustness of T.Thus, if an arbitrary statistic T generates sub-level sets satisfying assumption (22) then such statistic is g-robust.*

3.3 Dependence modeling: specialization vs. marginalization

Let $1 \le m < k, 1 \le i_1 < i_2 < \ldots < i_m \le k$. We assume that

$$\varphi_{m,Kt}(x_{i_1},\ldots,x_{i_m}) = C_{m,Kt}\left(\frac{|x_{i_1}|^{p_{i_1}}}{p_{i_1}}+\ldots+\frac{|x_{i_m}|^{p_{i_m}}}{p_{i_m}}\right)^{M-1} \cdot e^{-\beta\left(\frac{|x_{i_1}|^{p_{i_1}}}{p_{i_1}}+\ldots+\frac{|x_{i_m}|^{p_{i_m}}}{p_{i_m}}\right)}$$

and

$$\varphi_{m,PT7}(x_{i_1},\ldots,x_{i_m}) = C_{m,PT7}\left(1+\frac{1}{\nu}\left(\frac{|x_{i_1}|^{p_{i_1}}}{p_{i_1}}+\ldots+\frac{|x_{i_m}|^{p_{i_m}}}{p_{i_m}}\right)\right)^{-M}$$

are suitably normalized densities and call them m-dimensional specializations of the Kotz and Pearson type densities $\varphi^{(p)}_{Kt;M,\beta,\gamma}$ and $\varphi^{(p)}_{PT7;M,\nu}$, respectively. It is well known that marginal densities are not of the same type as specializations, in general. For the well known possibilities and problems of finding marginal densities of elliptically contoured distributions we refer to Fang et al. (1990).

Imagine now each of k experimenters observe another random variable, let them combine their (possibly dependent) observations by a vector and describe this vector by a joint cdf $F^{(k)}$ (possibly including dependence). In hindsight, did the experimenters construct the multivariate cdf $F^{(k)}$ starting from the specializations $F_1, ..., F_k$ of $F^{(k)}$ or from the marginal cdfs $F_1^*, ..., F_k^*$ of $F^{(k)}$? In other words, are the experimenters searching for a multivariate cdf $F^{(k)}$ such that their original observations follow marginal distributions or specializations of $F^{(k)}$? For an illustration, in the case $p_1 = ... = p_k$ and at hand of certain stock exchange indices, see Müller and Richter (2016).

A common way of studying dependence among components of random vectors makes use of marginal distributions and copulas. Here, we approach dependence by comparing a vector density with the product of all its one-dimensional specializations. To this end, let

$$c_{sp}(x) = J(x)/P(x), x \in \mathbb{R}^k$$

where $P(x) = \prod_{i=1}^{k} f_i(x_i)$ and J is the joint density which combines $f_1, \ldots, f_k$ by a certain dependence construction.

For comparison, let $x \to c^{(p)}(x)$, F_i and $g_i, i = 1, ..., k$ denote the Copula density, the marginal cdfs and pdfs of the distribution law $\Phi_g^{(p)}$, respectively. Then

$$c^{(p)}(F_1(x_1), ..., F_k(x_k)) = \varphi_g^{(p)}(x) / \prod_{i=1}^{k} g_i(x_i). \tag{24}$$

The following definition is therefore well motivated.

Definition 7 *We call c_{sp} the specialization copula density.*

Example 6 *Each of the functions*

$$f_i(x_i) = \frac{\gamma \beta^{\frac{1}{\gamma}(M-1+\frac{1}{p_i})}}{2p_i^{\frac{1}{p_i}-1} \Gamma\left(\frac{1}{\gamma}\left(M-1+\frac{1}{p_i}\right)\right)} \left(\frac{|x_i|^{p_i}}{p_i}\right)^{M-1} e^{-\beta\left(\frac{|x_i|^{p_i}}{p_i}\right)^{\gamma}}, \ i = 1, ..., k \tag{25}$$

is a one-dimensional specialization of the Kotz type density (17) meaning that, vice versa, $\varphi_{Kt;M,\beta,\gamma}$ in (17) generalizes f_i, $i = 1, \ldots, k$. In other words, the function in (17) is thought being build by a certain dependence construction applied to $f_1, ..., f_k$. Thus,

$$c_{sp,Kt}(x) = \varphi^{(p)}_{Kt;M,\beta,\gamma}(x) / \prod_{i=1}^{k} f_i(x_i) \ x \in \mathbb{R}^k.$$

and $\varphi^{(p)}_{Kt;M,\beta,\gamma}$ allows the representation

$$\varphi^{(p)}_{Kt;M,\beta,\gamma}(x) = c_{sp,Kt}(x) \prod_{i=1}^{k} f_i(x_i) \tag{26}$$

where the specialization copula density is explicitly given by

$$c_{sp,Kt}(x) = C \cdot \left(\frac{\sum_{i=1}^{k} \frac{|x_i|^{p_i}}{p_i}}{\prod_{i=1}^{k} \frac{|x_i|^{p_i}}{p_i}} \right)^{M-1} \frac{\exp\left\{\beta \sum_{i=1}^{k} \left(\frac{|x_i|^{p_i}}{p_i}\right)^{\gamma}\right\}}{\exp\left\{\beta \left(\sum_{i=1}^{k} \frac{|x_i|^{p_i}}{p_i}\right)^{\gamma}\right\}}$$

with

$$C = \frac{\beta^{\frac{(k-1)(1-M)}{\gamma}} \Gamma\left(\frac{M-1+\frac{1}{p_1}}{\gamma}\right) \cdot \ldots \cdot \Gamma\left(\frac{M-1+\frac{1}{p_k}}{\gamma}\right)}{\gamma^{k-1} B\left(\frac{1}{p_1}, ..., \frac{1}{p_k}\right) \Gamma\left(\frac{M-1+\frac{1}{p_1}+...+\frac{1}{p_k}}{\gamma}\right)}.$$

Clearly, searching for marginal densities is not as easy as determining specializations, here, and the corresponding copula density has not such nice structure.

Example 7 *Each of the functions*

$$f_i(x_i) = \frac{\Gamma(M)}{2\Gamma\left(1+\frac{1}{p_i}\right)\Gamma\left(M-\frac{1}{p_i}\right)(\nu p_i)^{\frac{1}{p_i}}}\left(1+\frac{|x_i|^{p_i}}{\nu p_i}\right)^{-M}$$

is a one-dimensional specialization of the Pearson Type VII pdf in (18). The pdf $\varphi^{(p)}_{PT7;M,\nu}$ allows the representation

$$\varphi^{(p)}_{PT7;M,\nu}(x) = c_{sp,PT7}(x) \prod_{i=1}^{k} f_i(x_i)$$

where the dependence function is

$$c_{sp,PT7}(x) = C \cdot \frac{\prod_{i=1}^{k}\left(1+\frac{|x_i|^{p_i}}{\nu p_i}\right)^{-M}}{\left(1+\sum_{i=1}^{k}\frac{|x_i|^{p_i}}{\nu p_i}\right)^{-M}}$$

with

$$C = p_1 \cdot \ldots \cdot p_k \frac{B\left(M-\frac{1}{p_1}, ..., M-\frac{1}{p_{k-1}}, M-\frac{1}{p_k}\right)}{B(\underbrace{M, ..., M}_{k-1}, M-\frac{1}{p_1}-...-\frac{1}{p_1})}.$$

As in the preceding example, the specialization copula density has a nice structure and is explicitly given.

Remark 10 *The univariate q-generalized normal distribution or q-power exponential distribution has been parameterized in different ways in the literature, for a recent survey see Dytso et al. (2018). For different purposes, any of these parameterizations can be used to derive modified representations of the distributions considered in this paper.*

4 Model extensions: a concluding remark

Although Definition 5 deals with the whole class of p-spherical distributions later consideration is concentrated on continuous p-spherical distributions. To finally widen again the view we refer to Remark 1 in Richter (2015a) where a way is described to derive new distributions from the elements of a given class of distributions by restricting the region of definition of such distributions. The following definition sums up that for the present situation.

Definition 8 *Let $\Sigma \in \mathfrak{B}(S_p)$ with $O_p(\Sigma) > 0$ and*

$$\omega_\Sigma(A) = \frac{O_p(A)}{O_p(\Sigma)}, A \in \mathfrak{B}(S_p) \cap \Sigma = \mathfrak{B}(\Sigma).$$

Then ω_Σ is called p-uniform distribution on $\mathfrak{B}(\Sigma)$.

The extension of the whole class of p-spherical distributions follows accordingly.

It might be a further task of future work to extend the class of q-spherical processes, $q > 0$, introduced in Müller and Richter (2019) to a class of p-spherical processes, $p = (p_1, ..., p_k) \in (0, \infty)^{\times k}$.

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