On Burr III Marshal Olkin family: development, properties, characterizations and applications

Fiaz Ahmad Bhatti[1*], G. G. Hamedani[2], Mustafa C. Korkmaz[3], Gauss M. Cordeiro[4], Haitham M. Yousof[5] and Munir Ahmad[1]

* Correspondence: fiazahmad72@
gmail.com
[1]National College of Business
Administration and Economics,
Lahore, Pakistan
Full list of author information is
available at the end of the article

Abstract

In this paper, a flexible family of distributions with unimodel, bimodal, increasing, increasing and decreasing, inverted bathtub and modified bathtub hazard rate called Burr III-Marshal Olkin-G (BIIIMO-G) family is developed on the basis of the T-X family technique. The density function of the BIIIMO-G family is arc, exponential, left- skewed, right-skewed and symmetrical shaped. Descriptive measures such as quantiles, moments, incomplete moments, inequality measures and reliability measures are theoretically established. The BIIIMO-G family is characterized via different techniques. Parameters of the BIIIMO-G family are estimated using maximum likelihood method. A simulation study is performed to illustrate the performance of the maximum likelihood estimates (MLEs). The potentiality of BIIIMO-G family is demonstrated by its application to real data sets.

Keywords: Moments, Reliability, Maximum likelihood Estimation, Characterizations

Introduction

Marshall and Olkin (1997) developed a new family of distributions with an additional shape parameter called Marshall and Olkin-G (MO-G) family. The survival function of MO-G family is

$$S(x) = \frac{\lambda \overline{G}(x, \psi)}{1 - \overline{\lambda} \overline{G}(x, \psi)}, \lambda > 0, \overline{\lambda} = 1 - \lambda, x \in \mathcal{R}, \tag{1}$$

where $G(x, \psi)$ is the baseline cumulative distribution function(cdf) which may depend on the vector parameter ψ.

Many famous MO-G families and its special distributions are available in literature such as Marshall-Olkin-G (Marshall and Olkin; 1997), the MO extended Lomax (Ghitany et al.; 2007), MO semi-Burr and MO Burr (Jayakumar and Mathew; 2008), MO q-Weibull (Jose et al.; 2010), MO extended Lindley (Ghitany et al.; 2012), the generalized MO-G (Nadarajah et al. (2013), the MO Fréchet (Krishna et al; 2013), the MO family (Cordeiro and Lemonte; 2013), MO extended Weibull(Santos-Neto et al.; 2014), the beta MO-G (Alizadeh et al. 2015), the MO generalized exponential (Ristić, & Kundu; 2015), MO gamma-Weibull (Saboor and Pogány; 2016), MO generalized-G

(Yousof et al.; 2018), MO additive Weibull (Afify et al.; 2018) and Weibull MO family (Korkmaz et al.; 2019).

This paper is sketched into the following sections. In Section 2, BIIIMO-G family is development via the T-X family technique. The basic structural properties and sub-models are also studied. In Section 3, two special models are studied. Section 4, deals with linear representations for the cdf and pdf of the BIIIMO-G family. In Section 5, moments, incomplete moments, inequality measures and some other properties are theoretically derived. In Section 6, stress-strength reliability and multicomponent stress-strength reliability of the model are studied. In Section 7, BIIIMO-G family is characterized via (i) conditional expectation; (ii) ratio of truncated moments and (iii) reverse hazard rate function. In Section 8, the maximum likelihood method is employed to estimate the parameters of the Burr III Marshall Olkin Weibull (BIIIMO-W) and Burr III Marshall Olkin Lindley (BIIIMO-L) distributions. In section 9, a simulation study is performed to illustrate the performance of the maximum likelihood estimates (MLEs). In Section 10, the potentiality of BIIIMO-G family is demonstrated by its application to real data sets: survival times of leukemia patients and bladder cancer patients' data. Goodness of fit of the probability distribution through different methods is studied. Section 11 contains concluding remarks.

Development of BIIIMO-G family

Alzaatreh et al. (2013) proposed a T-X family technique for the development of the wider families based on any probability density function (pdf). The cdf of the T-X family of distributions is given by

$$F(x) = \int_{a_1}^{W[G(x;\psi)]} r(t) \, dt, \qquad x \in \mathbb{R}, \tag{2}$$

where $r(t)$ is the pdf of a random variable (rv) T, where $T \in [a_1, a_2]$ for $-\infty \leq a_1 < a_2 < \infty$ and $W[G(x; \psi)]$ is a function of the baseline cumulative distribution function (cdf) of a rv X, depending on the vector parameter ψ and satisfies three conditions i) $W[G(x; \psi)] \in [a_1, a_2]$, ii) $W[G(x; \psi)]$ is differentiable and monotonically increasing and iii) $\lim_{x \to -\infty} W[G(x; \psi)] \to a_1$ and $\lim_{x \to \infty} [G(x; \psi)] \to a_2$. The pdf corresponding to (2) is

$$f(x) = \{\frac{\partial}{\partial x} W[G(x; \psi)]\} r\{W[G(x; \psi)]\}, \qquad x \in \mathbb{R}. \tag{3}$$

In this article, the BIIIMO-G family is developed via the T-X family technique by setting.

$$r(t) = \alpha\beta t^{-\beta-1}\{1 + t^{-\beta}\}^{-\alpha-1}, \; t > 0, \; \alpha > 0, \; \beta > 0, \; \text{and} \;\; W[G(x; \psi)] = -\log\left[\frac{\lambda\overline{G}(x,\psi)}{1-\overline{\lambda}\overline{G}(x,\psi)}\right].$$

Then, the cdf of BIIIMO-G family is

$$F(x; \alpha, \beta, \lambda, \psi) = \left\{1 + \left[-\log\left(\frac{\lambda\bar{G}(x; \psi)}{1-\bar{\lambda}\bar{G}(x; \psi)}\right)\right]^{-\beta}\right\}^{-\alpha}, x \in \mathbb{R}, \tag{4}$$

where $\alpha > 0$, $\beta > 0$, $\lambda > 0$ and $\psi > 0$ are parameters.

The pdf corresponding to (4) is given by

$$f(x; \alpha, \beta, \lambda, \psi) = \frac{\alpha \beta g(x; \psi)}{\bar{G}(x; \psi)(1 - \bar{\lambda} \bar{G}(x; \psi))} \left[-\log \left(\frac{\lambda \bar{G}(x; \psi)}{1 - \bar{\lambda} \bar{G}(x; \psi)} \right) \right]^{-\beta - 1} \times$$

$$\left\{ 1 + \left[-\log \left(\frac{\lambda \bar{G}(x; \psi)}{1 - \bar{\lambda} \bar{G}(x; \psi)} \right) \right]^{-\beta} \right\}^{-\alpha - 1}, x \in \mathbb{R}, \tag{5}$$

where $g(x; \psi)$ is the baseline pdf. In future, a rv with pdf (5) is denoted by $X \sim BIIIMO - G(\alpha, \beta, \lambda, \psi)$. The dependence on the parameter vector ψ can be omitted and simply write as $g(x) = g(x; \psi)$,

$G(x) = G(x; \psi)$ and $f(x) = f(x; \alpha, \beta, \lambda, \psi)$.

Let T be a BIII random variable with shape parameters α, β. The BIIIMO-G rv with cdf (4) can be obtained from

$$\Pr(X < x) = \Pr \left(T \leq -\log \left[\frac{\lambda \overline{G}(x)}{1 - \overline{\lambda} \overline{G}(x)} \right] \right).$$

Hence, the rv $X = G^{-1} [\frac{\lambda - \lambda \exp(-T)}{\lambda + \overline{\lambda} \exp(-T)}]$ has the BIIIMO-G distribution. The quantile function (qf) of X is the solution of the non-linear equation

$$x = Q(u) = Q_G \left(1 - \left\{ \overline{\lambda} + \lambda \exp \left[\left(u^{-\frac{1}{\alpha}} - 1 \right)^{-\frac{1}{\beta}} \right] \right\}^{-1} \right),$$

where $Q_G(.) = G^{-1}(.)$ is the qf of the baseline distribution. Hence, if U is a uniform rv on (0, 1), then $X = Q(U)$ follows the BIIIMO-G family.

Transformations and compounding

The BIIIMO-G family is derived through (i) ratio of the exponential and gamma random variables and (ii) compounding generalized inverse Weibull-MO (GIW-MO) and gamma distributions.

Lemma

i.   Let the random variable Z_1 have the exponential distribution with parameter value 1 and the random variable Z_2 have the fractional gamma i.e., $Z_2 \sim Gamma(\alpha, 1)$. Then, for

$$Z_1 = \left[-\log \left(\frac{\lambda \overline{G}(x)}{1 - \overline{\lambda} \overline{G}(x)} \right) \right]^{-\beta} Z_2,$$

we have

$$X = G^{-1} \left\{ \frac{\lambda + \lambda \exp \left[\left(\frac{Z_2}{Z_1} \right)^{\frac{1}{\beta}} \right]}{\overline{\lambda} + \lambda \exp \left[\left(\frac{Z_2}{Z_1} \right)^{\frac{1}{\beta}} \right]} \right\} \sim BIIIMO{-}G(\alpha, \beta, \lambda, \psi). \tag{6}$$

ii. If $Y|\beta, \lambda, \theta \sim GIWMO(y; \beta, \lambda, \theta)$ *and* $\theta|\alpha \sim gamma(\theta; \alpha)$, then integrating the effect of θ with the help of

$$f(y, \alpha, \beta, \lambda) = \int_0^\infty g(y|\beta, \lambda, \theta)g(\theta|\alpha)d\theta,$$

we have $Y \sim BIIIMO - G(\alpha, \beta, \lambda, \psi)..$

Structural properties of BIIIMO-G family

The survival, hazard, cumulative hazard, reverse hazard functions and the Mills ratio of a random variable X with BIIIMO-G family are, respectively given, by

$$S(x) = 1 - \left\{ 1 + \left[-\log\left(\frac{\lambda\overline{G}(x)}{1-\widetilde{\lambda G}(x)} \right) \right]^{-\beta} \right\}^{-\alpha}, \tag{7}$$

$$h(x) = \frac{\dfrac{\alpha\beta g(x)}{\overline{G}(x)\left(1-\widetilde{\lambda G}(x)\right)} \left[-\log\left(\frac{\lambda\overline{G}(x)}{1-\widetilde{\lambda G}(x)} \right) \right]^{-\beta-1}}{\left\{ 1 + \left[-\log\left(\frac{\lambda\overline{G}(x)}{1-\widetilde{\lambda G}(x)} \right) \right]^{-\beta} \right\}^{\alpha+1} - \left\{ 1 + \left[-\log\left(\frac{\lambda\overline{G}(x)}{1-\widetilde{\lambda G}(x)} \right) \right]^{-\beta} \right\}}, \tag{8}$$

$$r(x) = \frac{d}{dx} \ln\left\{ 1 + \left[-\log\left(\frac{\lambda\overline{G}(x)}{1-\widetilde{\lambda G}(x)} \right) \right]^{-\beta} \right\}^{-\alpha}, \tag{9}$$

$$H(x) = -\ln\left[1 - \left\{ 1 + \left[-\log\left(\frac{\lambda\overline{G}(x)}{1-\widetilde{\lambda G}(x)} \right) \right]^{-\beta} \right\}^{-\alpha} \right], \tag{10}$$

and

$$m(x) = \frac{\left\{ 1 + \left[-\log\left(\frac{\lambda\overline{G}(x)}{1-\widetilde{\lambda G}(x)} \right) \right]^{-\beta} \right\}^{\alpha+1} - \left\{ 1 + \left[-\log\left(\frac{\lambda\overline{G}(x)}{1-\widetilde{\lambda G}(x)} \right) \right]^{-\beta} \right\}}{\alpha\beta \dfrac{\lambda g(x)}{\overline{G}(x)\left(1-\widetilde{\lambda G}(x)\right)} \left[-\log\left(\frac{\lambda\overline{G}(x)}{1-\widetilde{\lambda G}(x)} \right) \right]^{-\beta-1}}. \tag{11}$$

The elasticity $e(x) = \frac{d\ln F(x)}{d \ln x} = xr(x)$ for BIIIMO-G family is

$$e(x) = \frac{d}{d \ln x} \ln\left\{ 1 + \left[-\log\left(\frac{\lambda\overline{G}(x)}{1-\widetilde{\lambda G}(x)} \right) \right]^{-\beta} \right\}^{-\alpha}. \tag{12}$$

The elasticity of BIIIMO-G family shows the behavior of the accumulation of probability in the domain of the random variable.

Sub-models

The BIIIMO-G family has the following sub models (Table 1).

Table 1 Sub-models of BIIIMO distribution

Variable	Parameters			Name of Distribution
X	α	β	λ	BIIIMO-G family
X	α	1	λ	Inverse Lomax MO-G family
X	1	β	λ	Log-logistic MO-G family
X	α	β	1	BIII-X family
X	α	1	1	Inverse Lomax-X family
X	1	β	1	Log-logistic-X family
X	1	β	λ	MO-G family
$\frac{1}{X}$	α	β	λ	BXIIMO-G family
X	1	1	1	Base line distribution

Special BIIIMO-G models

The BIIIMO-G family density (5) produces greater flexibility than any baseline distribution for data modeling. It can be most tractable, when the functions $g(x; \psi)$ and $G(x; \psi)$ have simple analytic expressions. Then, two special sub-models of BIIIMO-G family are introduced.

The BIIIMO-Weibull (BIIIMO-W) distribution

The cdf and pdf of the Weibull random variable are $g(x; b) = bx^{b-1}e^{-x^b}, x > 0, b > 0$ and $G(x; b) = 1 - e^{-x^b}, x \geq 0, b > 0$ where $\psi = b$. Then, the pdf of the BIIIMO-W model is given by
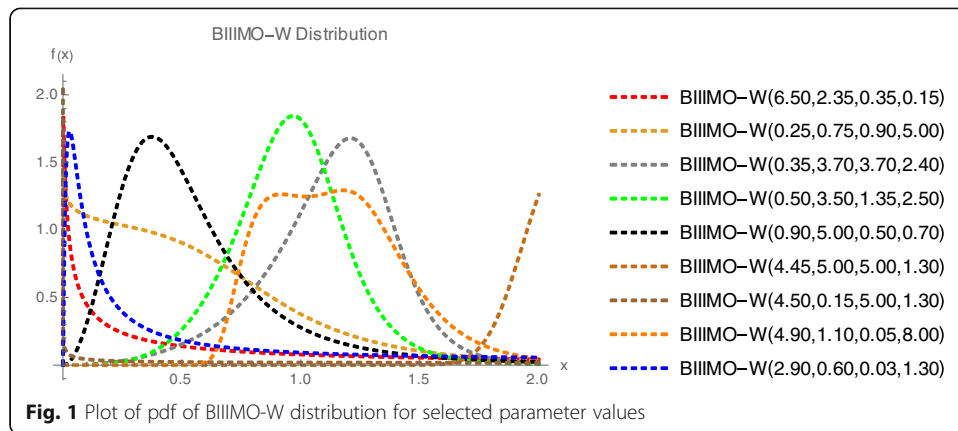
$$f(x; \alpha, \beta, \lambda, b) = \frac{b\alpha\beta x^{b-1}}{\left(1 - \overline{\lambda}e^{-x^b}\right)} \left[-\log\left(\frac{\lambda e^{-x^b}}{1 - \overline{\lambda}e^{-x^b}}\right) \right]^{-\beta-1} \times$$

$$\left\{ 1 + \gamma \left[-\log\left(\frac{\lambda e^{-x^b}}{1 - \overline{\lambda}e^{-x^b}}\right) \right]^{-\beta} \right\}^{-\alpha-1}, x > 0,$$

The following graphs show that shapes of BIIIMO-W density are arc, bimodal, exponential, left- skewed, right-skewed and symmetrical (Fig. 1). The BIIIMO-W distribution has uni-model, bimodal, increasing, increasing and decreasing, inverted bathtub and modified bathtub hazard rate function (hrf) Fig. 2.

The Burr III Marshal Olkin-Lindley (BIIIMO-L) distribution

The cdf and pdf of the Lindley random variable are $g(x; \psi) = \frac{b^2}{(1+b)}(1+x)e^{-bx}, x > 0, b > 0$ and $G(x; \psi) = 1 - (1 + \frac{bx}{1+b})e^{-bx}, x \geq 0$ where $\psi = b$. Then, the pdf of the BIIIMO-L model are given by

$$f(x; \alpha, \beta, \lambda, b) = \frac{\alpha\beta \frac{b^2}{(1+b)}(1+x)e^{-bx}}{\left(1 + \frac{bx}{1+b}\right)e^{-bx}} \left\{ -\log\left[\frac{\lambda\left(1 + \frac{bx}{1+b}\right)e^{-bx}}{1 - \overline{\lambda}\left(1 + \frac{bx}{1+b}\right)e^{-bx}} \right] \right\}^{-\beta-1} \times$$

$$\left(1 + \left\{ -\log\left[\frac{\lambda\left(1 + \frac{bx}{1+b}\right)e^{-bx}}{1 - \overline{\lambda}\left(1 + \frac{bx}{1+b}\right)e^{-bx}} \right] \right\}^{-\beta} \right)^{-\alpha-1}, x > 0,$$

Fig. 1 Plot of pdf of BIIIMO-W distribution for selected parameter values

The following graphs show that shapes of BIIIMO-L density are J, revers J, arc, exponential, left- skewed, right-skewed and symmetrical (Fig. 3). The BIIIMO-L distribution has unimodal, increasing, increasing and decreasing, decreasing-increasing-decreasing inverted bathtub, bathtub and modified bathtub hazard rate function (Fig. 4).
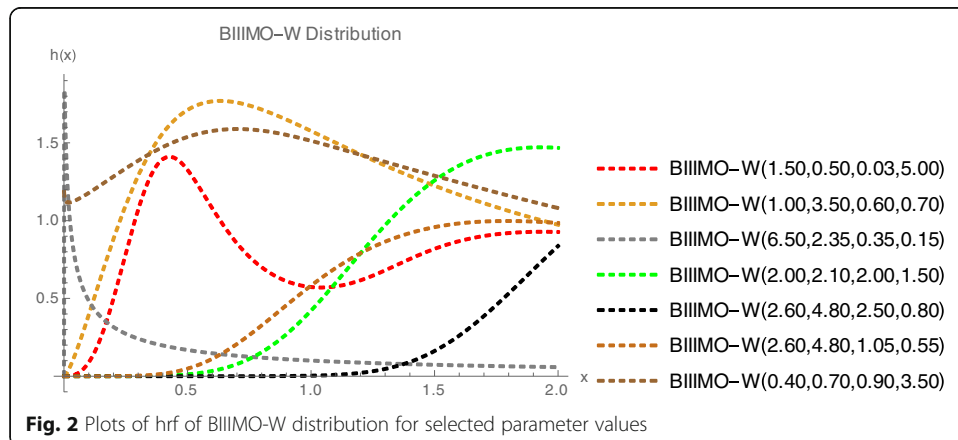
Useful expansions

In this sub-section, the linear representations for the cdf and pdf of the BIIIMO-G family are obtained. The cdf in (4) can be expressed as
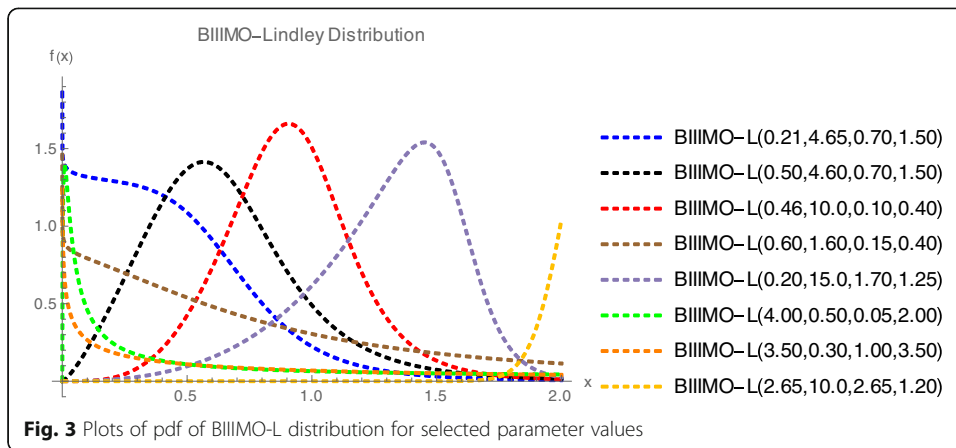
$$F(x) = 1 - \sum_{i \geq 1}^{\infty} (-1)^{i+1} \binom{\alpha+i-1}{i} \left[-\log\left(\frac{\lambda \overline{G}(x)}{1 - \overline{\lambda}\overline{G}(x)} \right) \right]^{-i\beta},$$

Following (Tahir et al. 2016), we have

$$\left[-\log\left(\frac{\lambda \overline{G}(x)}{1 - \overline{\lambda}\overline{G}(x)} \right) \right]^{-i\beta} = \frac{\sum_{k=0}^{\infty} p_k \left[1 - \left(\frac{\lambda \overline{G}(x)}{1 - \overline{\lambda}\overline{G}(x)} \right) \right]^{k}}{\sum_{k=0}^{\infty} q_k \left(1 - \frac{\lambda \overline{G}(x)}{1 - \overline{\lambda}\overline{G}(x)} \right)^{k}},$$

where

Fig. 2 Plots of hrf of BIIIMO-W distribution for selected parameter values

Fig. 3 Plots of pdf of BIIIMO-L distribution for selected parameter values

$$p_0 = 1,$$

$$p_1 = -\frac{1}{2} i\beta,$$

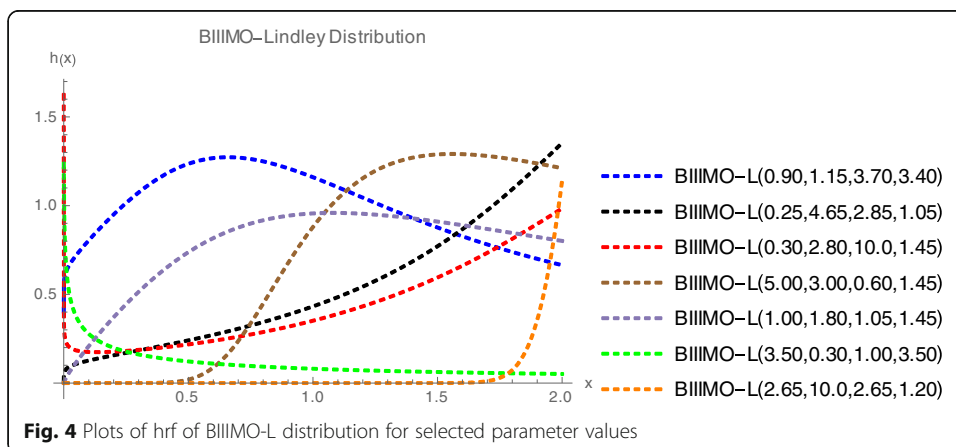$$p_2 = \frac{1}{24} \left[3(i\beta)^2 - 5i\beta \right],$$

$$p_3 = \frac{1}{48} \left[-(i\beta)^3 + 5(i\beta)^2 - 6i\beta \right],$$

$$p_4 = \frac{1}{5760} \left[15(i\beta)^4 - 150(i\beta)^3 + 48(i\beta)^2 - 502i\beta \right]$$

etc., and

$$q_k = q_k(i\beta) = \sum_{j=k}^{\infty} (-1)^{k+j} \binom{i\beta}{j} \binom{j}{k}$$

Then, we arrive at

Fig. 4 Plots of hrf of BIIIMO-L distribution for selected parameter values

$$F(x) = 1 - \sum_{i\geq 1}^{\infty} (-1)^{i+1} \binom{\alpha+i-1}{i} \frac{\sum_{k=0}^{\infty} p_k \left[1 - \left(\frac{\lambda \overline{G}(x)}{1-\widetilde{\lambda} \overline{G}(x)}\right)\right]^k}{\sum_{k=0}^{\infty} q_k \left(1 - \frac{\lambda \overline{G}(x)}{1-\widetilde{\lambda} \overline{G}(x)}\right)^k},$$

Using

$$\frac{\sum_{k=0}^{\infty} b_k x^k}{\sum_{k=0}^{\infty} a_k x^k} = \frac{1}{a_0} \sum_{k=0}^{\infty} c_k x^k,$$

where $c_n - \frac{1}{a_0} \sum_{k=1}^{n} c_{n-k} a_k - b_n = 0$ (see Gradshteyn and Ryzhik 2014),

we obtain $F(x) = 1 - \frac{\sum_{i\geq 1}^{\infty} (-1)^{1+i} \binom{\alpha+i-1}{i}}{q_0} \sum_{k=0}^{\infty} c_k \underbrace{\left[1 - \left(\frac{\lambda \overline{G}(x)}{1-\widetilde{\lambda} \overline{G}(x)}\right)\right]^k}_{A}.$

By applying the power series to the quantity A,

$$(1-z)^{k-1} = \sum_{h=0}^{\infty} (-1)^h \binom{k-1}{h} z^h,$$

where $|z| < 1$ and b is a real non-integer, we obtain

$$F(x) = 1 - \sum_{i\geq 1}^{\infty} (-1)^{1+i} \binom{\alpha+i-1}{i} \frac{1}{q_0} \left[c_0 + \sum_{k=1}^{\infty} c_k \sum_{h=1}^{k} (-1)^h \lambda^h \binom{k}{h} \overline{G}(x)^h \underbrace{\left[1-(1-\lambda)\overline{G}(x)\right]^{-h}}_{B}\right].$$

(13)

For $0 < \lambda \leq 1$, applying $(1-x)^{-n} = \sum_{\ell=0}^{\infty} \binom{n+\ell-1}{n-1} x^\ell$ to the quantity B, we arrive

$$F(x) = 1 - \sum_{i\geq 1}^{\infty} (-1)^{1+i} \binom{\alpha+i-1}{i} \frac{1}{q_0} \left[c_0 + \sum_{k=1}^{\infty} c_k \sum_{h=1}^{k} (-1)^h \binom{k}{h} \lambda^h \sum_{j=0}^{\infty} \binom{n+j-1}{n-1} (1-\lambda)^j \overline{G}(x)^{j+h}\right]$$

$$= 1 + \left[\begin{array}{l} \sum_{i\geq 1}^{\infty} \frac{c_0}{q_0} (-1)^{2+i} \binom{\alpha+i-1}{i} \\ + \sum_{j=0}^{\infty} \sum_{h=1}^{k} \sum_{k=1}^{\infty} \sum_{i\geq 1}^{\infty} \frac{c_k}{q_0} (-1)^{2+h+i} (1-\lambda)^j \lambda^h \binom{k}{h} \binom{n+j-1}{n-1} \binom{\alpha+i-1}{i} \overline{G}(x)^{j+h} \end{array}\right]$$

$$F(x) = 1 + \sum_{i\geq 1}^{\infty} v_i + \sum_{j=0}^{\infty} \sum_{h=1}^{k} \sum_{k=1}^{\infty} v_{j,h} \overline{H}_{j+h}(x),$$

(14)

where

$$v_{j,h} = \sum_{i\geq 1}^{\infty} \frac{c_k}{q_0} (-1)^{2+h+i} (1-\lambda)^j \lambda^h \binom{k}{h} \binom{n+j-1}{n-1} \binom{\alpha+i-1}{i}$$

$$v_i = \frac{c_0}{q_0} (-1)^{2+i} \binom{\alpha+i-1}{i}$$

and

$$\overline{H}_\theta(x) = \overline{G}(x)^\theta$$

is the cdf of the Lehmann type II Exp (1-G) model with power $\theta > 0$.

By differentiating (14) we get.

$$f(x) = \sum_{j=0}^{\infty} \sum_{h=1}^{k} \sum_{k=1}^{\infty} v_{j,h} h_{j+h}(x), \tag{15}$$

and

$$h_\theta(x) = \theta g(x) \overline{G}(x)^{\theta-1}$$

is the pdf of the Lehmann type II Exp (1-G) model with power $\theta > 0$. Equations (14) and (15) reveal that the BIIIMO-G density can be written as linear combinations of the Lehmann type II Exp(1-G) density functions. So, all properties of the new family can be derived based on the Lehmann type II Exp(1-G) density.

Moments

Moments, incomplete moments, inequality measures and some other properties are theoretically derived in this section.

Moments about origin

The rth moment of X, say μ'_r, follows from (15) as

$$\mu'_r = E(X^r) = \sum_{j=0}^{\infty} \sum_{h=1}^{k} \sum_{k=1}^{\infty} v_{j,h} E\left(Y^r_m\right) \tag{16}$$

Henceforth, Y_m denotes the Lehmann type **II** exp-(1-G) distribution with power parameter m.

The nth central moment of X, say M_n, is given by

$$M_n = E(X - \mu'_1)^n = \sum_{r=0}^{n} \sum_{j=0}^{\infty} \sum_{h=1}^{k} \sum_{k=1}^{\infty} v_{j,h} (-1)^{n-r} \binom{n}{r} \mu'^{(n-r)}_r E(Y^r_m) \tag{17}$$

The cumulants (κ_n) of X follow recursively from

$$\kappa_n = \mu'_n - \sum_{r=0}^{n-1} \binom{n-1}{r-1} \kappa_r \mu'_{n-r} \tag{18}$$

where $\kappa_1 = \mu'_1, \kappa_2 = \mu'_2 - \mu'^2_1, \kappa_3 = \mu'_3 - 3\mu'_2\mu'_1 + \mu'^3_1$, etc. The skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships.

Generating function

The moment generating function (mgf) $M_X(t) = E(e^{t X})$ of X is given by

$$M_X(t) = \sum_{j=0}^{\infty} \sum_{h=1}^{k} \sum_{k=1}^{\infty} v_{j,h} M_m(t), \tag{19}$$

where $M_m(t)$ is the mgf of Y_m. Hence, $M_X(t)$ can be determined from Lehmann type **II** exp-(1-G) generating function.

Incomplete moments

The sth incomplete moment, say $I_s(t)$, of X can be expressed from (15) as

$$I_s(t) = \int\limits_{-\infty}^{t} x^s f(x)dx = \sum\nolimits_{j=0}^{\infty}\sum\nolimits_{h=1}^{k}\sum\nolimits_{k=1}^{\infty} \mathsf{v}_{j,h} \int_{-\infty}^{t} x^s h_{j+h}(x)dx \tag{20}$$

A general formula for the first incomplete moment, $I_1(t)$, can be derived from the last equation (with $s = 1$).

The first incomplete moment can be applied to construct Bonferroni and Lorenz curves defined for a given probability π by $B(\pi) = I_1(q)/(\pi\mu_1^{'})$ and $L(\pi) = I_1(q)/\mu_1^{'}$, respectively, where $\mu_1^{'} = E(X)$ and $q = Q(\pi)$ is the qf of X at π. The mean deviations about the mean $[\delta_1 = E(|X-\mu_1^{'}|)]$ and about the median $[\delta_2 = E(|X - M|)]$ of X are given by $\delta_1 = 2\mu_1^{'} F(\mu_1^{'})-2I_1(\mu_1^{'})$ and $\delta_2 = \mu_1^{'}-2I_1(M)$, respectively, where $\mu_1^{'} = E(X)$, $M = Median$ $(X) = Q(\frac{1}{2})$ is the median and $F(\mu_1^{'})$ is easily calculated from (4).

Table 2 shows the numerical measures of the median, mean, standard deviation, skewness and Kurtosis of the BIIIMO-W distribution for selected parameter values to describe their effect on these measures.

Table 3 shows the numerical measures of the median, mean, standard deviation, skewness and Kurtosis of the BIIIMO-L distribution for selected parameter values to describe their effect on these measures.

Reliability measures

In this section, different reliability measures for the BIIIMO-G family are studied.

Stress-strength reliability of BIIIMO-G family

Let $X_1 \sim BIIIMO - G(\alpha_1, \beta, \lambda, \psi)$, $X_2 \sim BIIIMO - G(\alpha_2, \beta, \lambda, \psi)$ and X_1 represents strength and X_2 represents stress. Then, the reliability of the component is:

$$R = \Pr(X_2 < X_1) = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{x_1} f(x_1, x_2)dx_2 dx_1 = \int\limits_{0}^{\infty} f_{x_1}(x)F_{x_2}(x)dx,$$

$$R = \int\limits_{0}^{\infty} \frac{\alpha_1\beta g(x)}{\overline{G}(x)\left(1-\overline{\lambda}\overline{G}(x)\right)} \left[-\log\left(\frac{\lambda\overline{G}(x)}{1-\overline{\lambda}\overline{G}(x)}\right)\right]^{-\beta-1} \left\{1+\left[-\log\left(\frac{\lambda\overline{G}(x)}{1-\overline{\lambda}\overline{G}(x)}\right)\right]^{-\beta}\right\}^{-\alpha_1-1}$$

$$\left\{1+\left[-\log\left(\frac{\lambda\overline{G}(x)}{1-\overline{\lambda}\overline{G}(x)}\right)\right]^{-\beta}\right\}^{-\alpha_2} dx = \frac{\alpha_1}{(\alpha_1 + \alpha_2)}$$

$$\tag{21}$$

Therefore R is independent of β, λ *and* ψ.

Multicomponent stress-strength reliability estimator $R_{s,\,\kappa}$ based on BIIIMO-G family

Suppose a machine has at least "s" components working out of " κ " components. The strengths of all components of system are X_1, X_2, ...X_κ and stress Y is applied to the system. Both the strengths X_1, X_2, ...X_κ are i.i.d. and are independent of stress Y. F

Table 2 Median, mean, standard deviation, skewness and Kurtosis of the BIIIMO-W distribution

Parameters *a, β, λ, b*	Median	Mean	Standard Deviation	Skewness	Kurtosis
1,5,5,5	1.1771	1.1780	0.0545	0.2111	4.9161
2,5,5,5	1.2034	1.2079	0.04693	0.8716	5.9434
3,5,5,5	1.2174	1.2232	0.0452	1.1058	6.6793
5,0.1,5,5	1.7972	1.8793	0.9214	0.1150	1.9890
5,0.5,5,5	1.9479	2.0645	0.6437	0.5428	2.4137
5,0.6,5,5	1.8321	1.9746	0.6159	0.8302	2.9991
5,1,5,5	1.5246	1.6301	0.3962	1.7014	6.7042
0.5,1.5,0.5,5	0.8426	1.3586	1.1866	1.4186	3.2418
0.5,1.5,1.5,5	0.9185	0.9232	0.2749	0.6282	5.9778
0.5,1.5,2.5,5	0.9858	0.9805	0.2668	0.4229	5.6954
0.1,3.25,0.5,5	0.5710	0.5575	0.2477	−0.0171	2.9788
0.1,3.25,0.75,5	0.6175	0.5985	0.2616	−0.0403	3.5046
0.1,3.25,1,5	0.6521	0.6282	0.2694	−0.1980	2.2918
1,10,1,10	1.0000	1.0001	0.0181	0.0954	4.2672
5,5,5,1	2.8679	2.9859	0.5965	3.9302	127.836
5,5,5,3	1.4208	1.4346	0.0866	1.4729	9.1049
5,5,5,5	1.2346	1.2413	0.0443	1.3008	7.4790

Table 3 Median, mean, standard deviation, skewness and Kurtosis of the BIIIMO-Lindley Distribution

Parameters *a, β, λ, b*	Median	Mean	Standard Deviation	Skewness	Kurtosis
1,1,1,1	1.5675	5.3348	15.3851	8.2183	90.6069
2,1,1,1	3.3663	9.5872	22.6103	6.4326	55.7313
3,1,1,1	5.0424	13.5865	29.5541	5.8056	45.5406
4,1,1,1	6.6589	17.6427	37.3102	5.5979	42.3776
5,1,1,1	8.1887	19.9071	36.5353	4.6082	29.1756
1,10,1,1	1.5826	1.6022	0.2585	0.8210	6.4705
0.65,10,5,1	3.1084	3.1032	0.3673	0.0413	4.5913
0.65,10,5,0.15	23.9115	23.8698	2.5003	0.0084	4.4424
0.5,10,3,0.10	29.9001	29.7078	3.9877	−0.1783	4.1985
0.15,20,0.5,0.95	0.8711	0.8362	0.2341	−0.5213	2.9375
0.15,20,0.5,0.5	1.9486	1.8711	0.4854	−0.5968	3.0644
0.5,20,0.5,0.5	2.3013	2.2875	0.2677	−0.2134	4.1862
0.1,9,0.5,0.5	1.1787	1.2283	0.7764	0.4073	2.6804
0.1,9,0.35,0.5	0.8983	0.9548	0.6292	0.5181	2.9726
0.1,10,1.1,0.4	2.8248	2.7678	1.4233	0.0055	2.2482
0.1,10,1.3,0.4	3.1227	3.0411	1.5263	−0.0426	2.2369
0.1,10,2,0.4	3.9758	3.8169	1.7934	−0.1728	2.2366
0.1,10,2,1.8	0.6578	0.6423	0.3405	−0.0045	2.1782
0.1,10,2,1.9	0.6160	0.6020	0.3203	0.0051	2.1889

and G are the cdf of X and Y respectively. The reliability of a machine is the probability that the machine functions properly.

Let $X \sim BIIIMO - G(\alpha_1, \beta, \lambda, \psi)$, $Y \sim BIIIMO - G(\alpha_2, \beta, \lambda, \psi)$ with common parameters β, λ, ψ and unknown shape parameters α_1 *and* α_2. The multicomponent stress- strength reliability for BIIIMO-G family is given by (Bhattacharyya and Johnson 1974).

$$R_{s,m} = P(strengths > stress) = P[at\,least\,"s"\,of\,(X_1, X_2, ...X_\kappa)\,exceed\,Y], \qquad (22)$$

$$R_{s,\kappa} = \sum_{l=s}^{\kappa} \binom{\kappa}{l} \int_0^\infty [1-F(y)]^l\,[F(y)]^{\kappa-l}dG(y) \qquad (23)$$

$$R_{s,\kappa} = \sum_{\ell=s}^{\kappa} \binom{\kappa}{\ell} \left[\int_0^\infty \frac{\left(1-\left\{1+\left[-\log\left(\frac{\lambda\overline{G}(x)}{1-\tilde{\lambda}\overline{G}(x)}\right)\right]^{-\beta}\right\}^{-\alpha_1}\right)^\ell \left(\left\{1+\left[-\log\left(\frac{\lambda\overline{G}(x)}{1-\tilde{\lambda}\overline{G}(x)}\right)\right]^{-\beta}\right\}^{-\alpha_1}\right)^{(\kappa-\ell)}}{\frac{\alpha_2\beta g(x)}{\overline{G}(x)\left(1-\tilde{\lambda}\overline{G}(x)\right)}\left[-\log\left(\frac{\lambda\overline{G}(x)}{1-\tilde{\lambda}\overline{G}(x)}\right)\right]^{-\beta-1}\left\{1+\left[-\log\left(\frac{\lambda\overline{G}(x)}{1-\tilde{\lambda}\overline{G}(x)}\right)\right]^{-\beta}\right\}^{-\alpha_2-1}} \times \right] dx$$

Let $t = \left\{1 + \left[-\log(\frac{\lambda\overline{G}(x)}{1-\tilde{\lambda}\overline{G}(x)})\right]^{-\beta}\right\}^{-\alpha_2}$, then we obtain

$$R_{s,\kappa} = \sum_{\ell=s}^{\kappa} \binom{\kappa}{\ell} \int_0^1 (1-t^\nu)^\ell\,t^{\nu(\kappa-\ell)}dt.$$

Let $z = t^\nu, t = z^{\frac{1}{\nu}},\ dt = \frac{1}{\nu}z^{\frac{1}{\nu}-1}dz$

, then

$$R_{s,\kappa} = \sum_{\ell=s}^{\kappa} \binom{\kappa}{\ell} \int_0^1 (1-t^\nu)^\ell\,t^{\nu(\kappa-\ell)}dt,$$

$$R_{s,\kappa} = \sum_{\ell=s}^{\kappa} \binom{\kappa}{\ell} \int_0^1 (1-z)^\ell\,z^{(\kappa-\ell)}\frac{1}{\nu}z^{\frac{1}{\nu}-1}dz,$$

$$R_{s,\kappa} = \frac{1}{\nu}\sum_{\ell=s}^{\kappa} \binom{\kappa}{\ell} B\left(1+\ell, \kappa-\ell+\frac{1}{\nu}\right).,$$

$$R_{s,\kappa} = \frac{1}{\nu}\sum_{\ell=s}^{\kappa} \frac{\kappa!}{(\kappa-\ell)!}\left[\prod_{j=0}^\ell \left(\kappa-j+\frac{1}{\nu}\right)\right]^{-1}\,where\,\nu = \frac{\alpha_2}{\alpha_1} \qquad (24)$$

The probability $R_{s,\ \kappa}$ in (24) is called multicomponent stress-strength model reliability.

Characterizations

In this section, BIIIMO-G family is characterized via: (i) conditional expectation; (ii) ratio of truncated moments and (iii) reverse hazard rate function.

Characterization based on conditional expectation

Here BIIIMO-G family is characterized via conditional expectation.

Proposition

Let $X : \Omega \to (0, \infty)$ be a continuous random variable with cdf $F(x)$
($0 < F(x) < 1$ *for* $x \geq 0$), then for $\alpha > 1$, X has cdf (4) if and only if

$$E\left\{\left[-\log\left(\frac{\lambda\overline{G}(X)}{1-\overline{\lambda}\overline{G}(X)}\right)\right]^{-\beta}\middle| X < t\right\} = \frac{1}{(\alpha-1)}\left\{1+\alpha\left[-\log\left(\frac{\lambda\overline{G}(t)}{1-\overline{\lambda}\overline{G}(t)}\right)\right]^{-\beta}\right\}, \quad for\ t > 0 \qquad (25)$$

Proof. If pdf of X is (5), then

$$E\left\{\left[-\log\left(\frac{\lambda\overline{G}(X)}{1-\overline{\lambda}\overline{G}(X)}\right)\right]^{-\beta}\middle| X < t\right\} = (F(t))^{-1}\int_0^t\left[-\log\left(\frac{\lambda\overline{G}(x)}{1-\overline{\lambda}\overline{G}(x)}\right)\right]^{-\beta}f(x)dx$$

$$= (F(t))^{-1}\int_0^t\left(\begin{array}{c}[-\log(\frac{\lambda\overline{G}(x)}{1-\overline{\lambda}\overline{G}(x)})]^{-\beta}\alpha\beta\frac{g(x)}{\overline{G}(x)(1-\overline{\lambda}\overline{G}(x))}\times \\ [-\log(\frac{\lambda\overline{G}(x)}{1-\overline{\lambda}\overline{G}(x)})]^{-\beta-1}\{1+[-\log(\frac{\lambda\overline{G}(x)}{1-\overline{\lambda}\overline{G}(x)})]^{-\beta}\}^{-\alpha-1}\end{array}\right)dx$$

Upon integration by parts and simplification, we obtain

$$E\left\{\left[-\log\left(\frac{\lambda\overline{G}(X)}{1-\overline{\lambda}\overline{G}(X)}\right)\right]^{-\beta}\middle| X < t\right\} = \frac{1}{(\alpha-1)}\left\{1+\alpha\left[-\log\left(\frac{\lambda\overline{G}(t)}{1-\overline{\lambda}\overline{G}(t)}\right)\right]^{-\beta}\right\}, \quad t > 0.$$

Conversely, if (25) holds, then

$$\int_0^t\left[-\log\left(\frac{\lambda\overline{G}(x)}{1-\overline{\lambda}\overline{G}(x)}\right)\right]^{-\beta}f(x)dx = \frac{F(t)}{(\alpha-1)}\left\{1+\alpha\left[-\log\left(\frac{\lambda\overline{G}(t)}{1-\overline{\lambda}\overline{G}(t)}\right)\right]^{-\beta}\right\} \qquad (26)$$

Differentiating (26) with respect to t, we obtain.

$$[-\log(\tfrac{\lambda\overline{G}(t)}{1-\overline{\lambda}\overline{G}(t)})]^{-\beta}f(t) = \tfrac{f(t)}{(\alpha-1)}\{1+\alpha[-\log(\tfrac{\lambda\overline{G}(t)}{1-\overline{\lambda}\overline{G}(t)})]^{-\beta}\} - \tfrac{F(t)}{(\alpha-1)}[\tfrac{\alpha\beta g(t)}{\overline{G}(x)(1-\overline{\lambda}\overline{G}(t))}[-\log(\tfrac{\lambda\overline{G}(t)}{1-\overline{\lambda}\overline{G}(t)})]^{-\beta-1}]$$

After simplification and integration we arrive at

$$F(t) = \left\{1+\left[-\log\left(\frac{\lambda\overline{G}(t)}{1-\overline{\lambda}\overline{G}(t)}\right)\right]^{-\beta}\right\}^{-\alpha}, t\geq 0$$

Characterization of BIIIMO-G family through ratio of truncated moments

Here BIIIMO-G family is characterized using Theorem G (Glänzel; 1987) on the basis of a simple relationship between two truncated moments of X.

Proposition

Let $X : \Omega \to \mathbb{R}$ be a continuous random variable.

Let $h_1(x) = \frac{1}{\alpha}\left\{1+\left[-\log\left(\frac{\lambda\overline{G}(x)}{1-\overline{\lambda}\overline{G}(x)}\right)\right]^{-\beta}\right\}^{\alpha+1}, x\in\mathbb{R}$

and

$$h_2(x) = \frac{2\left\{1 + \left[-\log\left(\frac{\lambda\overline{G}(x)}{1-\widetilde{\lambda}\overline{G}(x)}\right)\right]^{-\beta}\right\}^{\alpha+1}}{\alpha\left[-\log\left(\frac{\lambda\overline{G}(x)}{1-\widetilde{\lambda}\overline{G}(x)}\right)\right]^{\beta}}, x\in\mathbb{R}$$

The random variable X has pdf (5) if and only if that the function $p(x)$ (defined in theorem G) has the form $p(x) = \left[-\log\left(\frac{\lambda\overline{G}(x)}{1-\widetilde{\lambda}\overline{G}(x)}\right)\right]^{\beta}$, $x\in\mathbb{R}$.

Proof. For random variable X with pdf (5),

$$(1-F(x))E(h_1(X)|X\geq x) = \left[-\log\left(\frac{\lambda\overline{G}(x)}{1-\widetilde{\lambda}\overline{G}(x)}\right)\right]^{-\beta}$$

and

$$(1-F(x))E(h_2(X)|X\geq x) = \left[-\log\left(\frac{\lambda\overline{G}(x)}{1-\widetilde{\lambda}\overline{G}(x)}\right)\right]^{-2\beta}, x\in\mathbb{R}.$$

$$\frac{E[h_1(x)|X\geq x]}{E[h_2(x)|X\geq x]} = p(x) = \left[-\log\left(\frac{\lambda\overline{G}(x)}{1-\widetilde{\lambda}\overline{G}(x)}\right)\right]^{\beta}, x\in\mathbb{R},$$

and

$$p'(x) = \frac{\beta g(x)}{\overline{G}(x)(1-\widetilde{\lambda}\overline{G}(x))}\left[-\log\left(\frac{\lambda\overline{G}(x)}{1-\widetilde{\lambda}\overline{G}(x)}\right)\right]^{\beta-1}, x\in\mathbb{R}.$$

The differential equation

$$s'(x) = \frac{p'(x)h_2(x)}{p(x)h_2(x)-h_1(x)} = \frac{2\beta g(t)}{\overline{G}(x)\left(1-\widetilde{\lambda}\overline{G}(t)\right)}\left[-\log\left(\frac{\lambda\overline{G}(x)}{1-\widetilde{\lambda}\overline{G}(x)}\right)\right]^{-1}$$

has solution

$$s(x) = \ln\left[-\log\left(\frac{\lambda\overline{G}(x)}{1-\widetilde{\lambda}\overline{G}(x)}\right)\right]^{2\beta}, x\in\mathbb{R}.$$

Therefore, in the light of theorem G, X has pdf (5).

Corollary

Let $X:\Omega\to\mathbb{R}$ be a continuous random variable and let.

$$h_2(x) = \frac{2\left\{1+\left[-\log\left(\frac{\lambda\overline{G}(x)}{1-\widetilde{\lambda}\overline{G}(x)}\right)\right]^{-\beta}\right\}^{\alpha+1}}{\alpha\left[-\log\left(\frac{\lambda\overline{G}(x)}{1-\widetilde{\lambda}\overline{G}(x)}\right)\right]^{\beta}}, x\in\mathbb{R}.$$ The pdf of X is (5) if and only if there exist

functions

$p(x)$ *and* $h_1(x)$ defined in Theorem G, satisfying the differential equation

$$\frac{p'(x)}{p(x)h_2(x)-h_1(x)} = \frac{\alpha\beta g(x)}{\overline{G}(x)\left(1-\widetilde{\lambda}\overline{G}(x)\right)}\left[-\log\left(\frac{\lambda\overline{G}(x)}{1-\widetilde{\lambda}\overline{G}(x)}\right)\right]^{\beta-1}\left\{1+\left[-\log\left(\frac{\lambda\overline{G}(x)}{1-\widetilde{\lambda}\overline{G}(x)}\right)\right]^{-\beta}\right\}^{-\alpha-1}.$$

$$(27)$$

Remark

The solution of (27) is

$$p(x) = \left[-\log\left(\frac{\lambda\overline{G}(x)}{1-\overline{\lambda}\overline{G}(x)}\right)\right]^{2\beta}\int\left[\begin{array}{l}-h_1(x)\dfrac{\alpha\beta g(x)}{\overline{G}(x)\left(1-\overline{\lambda}\overline{G}(x)\right)}\left[-\log\left(\dfrac{\lambda\overline{G}(x)}{1-\overline{\lambda}\overline{G}(x)}\right)\right]^{-\beta-1}\times\\[4mm]\left\{1+\left[-\log\left(\dfrac{\lambda\overline{G}(x)}{1-\overline{\lambda}\overline{G}(x)}\right)\right]^{-\beta}\right\}^{-\alpha-1}dx\end{array}\right]+D,$$

where D is a constant.

Characterization of BIIIMO-G family via reverse Hazard rate function

Definition

Let $X:\Omega\to(0,\infty)$ be a continuous random variable with cdf $F(x)$ and pdf $f(x)$. The reverse hazard function, r_F, of a twice differentiable distribution function F, satisfies the differential equation

$$\frac{d}{dx}[\ln f(x)] = \frac{r'_F(x)}{r_F(x)} + r_F(x). \tag{28}$$

Proposition

Let $X:\Omega\to(0,\infty)$ be continuous random variable. The pdf of X is (5) if and only if its reverse hazard function, r_F satisfies the first order differential equation

$$r'_F(x)\left(1+\left\{-\log\left[\frac{\lambda\overline{G}(x)}{1-\overline{\lambda}\overline{G}(x)}\right]\right\}^{-\beta}\right) - \frac{\beta g(x)\left[-\log\left(\frac{\lambda\overline{G}(x)}{1-\overline{\lambda}\overline{G}(x)}\right)\right]^{-\beta-1}}{\overline{G}(x)\left(1-\overline{\lambda}\overline{G}(x)\right)}r_F(x)$$
$$= \alpha\beta\frac{d}{dx}\left\{\frac{g(x)\left[-\log\left(\frac{\lambda\overline{G}(x)}{1-\overline{\lambda}\overline{G}(x)}\right)\right]^{-\beta-1}}{\overline{G}(x)\left(1-\overline{\lambda}\overline{G}(x)\right)}\right\}. \tag{29}$$

Proof If

X has pdf (5), then (29) holds. Now if (29) holds, then

$$\frac{d}{dx}\left[r_F(x)\left(1+\left\{-\log\left[\frac{\lambda\overline{G}(x)}{1-\overline{\lambda}\overline{G}(x)}\right]\right\}^{-\beta}\right)\right] = \alpha\beta\frac{d}{dx}\left\{\frac{g(x)}{\overline{G}(x)\left(1-\overline{\lambda}G(x)\right)}\left[-\log\left(\frac{\lambda\overline{G}(x)}{1-\overline{\lambda}\overline{G}(x)}\right)\right]^{-\beta-1}\right\},$$

or

$$r_F(x) = \frac{\alpha\beta g(x)}{\overline{G}(x)\left(1-\overline{\lambda}\overline{G}(x)\right)}\left[-\log\left(\frac{\lambda\overline{G}(x)}{1-\overline{\lambda}\overline{G}(x)}\right)\right]^{-\beta-1}\left\{1+\left[-\log\left(\frac{\lambda\overline{G}(x)}{1-\overline{\lambda}\overline{G}(x)}\right)\right]^{-\beta}\right\}^{-1}, x\in\mathbb{R},$$

which is the reverse hazard function of the BIIIMO-G Family.

Maximum likelihood estimation

In this section, parameter estimates are derived using maximum likelihood method. The log-likelihood function for the vector of parameters $\Phi=(\alpha,\beta,\lambda,\psi^\tau)$ of BIIIMO-G family is

$$\ell(\Phi) = \ln L(\Phi)$$

$$= \begin{pmatrix} n \ln(\alpha) + n \ln(\beta) + \sum_{i=1}^{n} \ln g(x_i; \psi) - \sum_{i=1}^{n} \ln \overline{G}(x_i; \psi) - \\ \sum_{i=1}^{n} \ln \left(1 - \overline{\lambda G}(x_i; \psi)\right) - (\beta + 1) \sum_{i=1}^{n} \ln \left[-\log\left(\frac{\lambda \overline{G}(x_i; \psi)}{1 - \overline{\lambda G}(x_i; \psi)}\right) \right] - \\ (\alpha + 1) \sum_{i=1}^{n} \ln \left\{ 1 + \left[-\log\left(\frac{\lambda \overline{G}(x_i; \psi)}{1 - \overline{\overline{\lambda G}}(x_i; \psi)}\right) \right]^{-\beta} \right\} \end{pmatrix} \quad (30)$$

In order to compute the estimates of the parameters α, β, λ, ψ^τ, the following nonlinear equations must be solved simultaneously:

$$\frac{\delta}{\delta\alpha}\ell(\Phi) = \frac{n}{\alpha} - \sum_{i=1}^{n} \ln\left\{ 1 + \left[-\log\left(\frac{\lambda \overline{G}(x_i; \psi)}{1 - \overline{\lambda G}(x_i; \psi)}\right) \right]^{-\beta} \right\} = 0, \quad (31)$$

$$\frac{\delta}{\delta\beta}\ell(\Phi) = \frac{n}{\beta} - \sum_{i=1}^{n} \ln\left[-\log\left(\frac{\lambda \overline{G}(x_i; \psi)}{1 - \overline{\lambda G}(x_i; \psi)}\right) \right] + (\alpha + 1) \sum_{i=1}^{n} \ln\left[-\log\left(\frac{\lambda \overline{G}(x_i; \psi)}{1 - \overline{\lambda G}(x_i; \psi)}\right) \right] \left\{ \left[-\log\left(\frac{\lambda \overline{G}(x_i; \psi)}{1 - \overline{\lambda G}(x_i; \psi)}\right) \right]^{\beta} + 1 \right\}^{-1},$$

$$(32)$$

$$\frac{\delta}{\delta\psi}\ell(\Phi) = \begin{pmatrix} \sum_{i=1}^{n} \frac{g'(x_i; \psi)}{g(x_i; \psi)} + \sum_{i=1}^{n} \frac{g'(x_i; \psi)}{\overline{G}(x_i; \psi)} - \sum_{i=1}^{n} \frac{\overline{\lambda} g'(x_i; \psi)}{[1 - \overline{\lambda G}(x_i; \psi)]} \\ -(\beta + 1) \sum_{i=1}^{n} \frac{g'(x_i; \psi) \left[-\log\left(\frac{\lambda \overline{G}(x_i; \psi)}{1 - \overline{\lambda G}(x_i; \psi)}\right) \right]^{-1}}{\left(\overline{G}(x_i; \psi) - \overline{\lambda G}^2(x_i; \psi) \right)} \\ -(\alpha + 1)\beta \sum_{i=1}^{n} \left\{ 1 + \left[-\log\left(\frac{\lambda \overline{G}(x_i; \psi)}{1 - \overline{\lambda G}(x_i; \psi)}\right) \right]^{\beta} \right\} \frac{g'(x_i; \psi) \left[-\log\left(\frac{\lambda \overline{G}(x_i; \psi)}{1 - \overline{\lambda G}(x_i; \psi)}\right) \right]^{\beta-1}}{\left(\overline{G}(x_i; \psi) - \overline{\lambda G}^2(x_i; \psi) \right)} \end{pmatrix} = 0.$$

$$(33)$$

where $g'(x_i; \psi) = \frac{\delta}{\delta\psi} g(x_i; \psi)$.

Simulation studies

In this Section, we perform the simulation study to see the performance of MLE's of BIIIMO-L distribution. The random number generation is obtained with inverse of its cdf. The MLEs, say $(\hat{\alpha}_i, \hat{\beta}_i, \hat{\lambda}_i, \hat{b}_i)$ for i = 1,2,…,N, have been obtained by CG routine in R programme.

The simulation study is based on graphical results. We generate $N = 1000$ samples of sizes $n = 20,30,…,1000$ from BIIIMOL distribution and get true values of $\alpha = 1.5$, $\beta = 2.55$, $\lambda = 0.095$ and $b = 5$ for this simulation study. We also calculate the mean, standard deviations (sd), bias and mean square error (MSE) of the MLEs. The bias and MSE are calculated by (for $h = \alpha, \beta, \lambda, b$)

$$Bias_h = \frac{1}{N} \sum_{i=1}^{N} \left(\hat{h}_i - h \right)$$

and

$$MSE_h = \frac{1}{N} \sum_{i=1}^{N} \left(\hat{h}_i - h \right)^2.$$

The results are given by Fig. 5. Figure 5 reveals that the empirical means tend to the true parameter values and that the sds, biases and MSEs decrease when the sample size

increases as expected. These results are in agreement with first-order asymptotic theory.

Applications

The BIIIMO-W and BIIIMO-L distributions are compared with sub- and competing models. Different goodness fit measures such Cramer-von Mises (W*), Anderson Darling (A*), Kolmogorov- Smirnov statistics with *p*-values [K-S(p-values], Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC) and likelihood ratio statistics are computed for survival times of leukemia patients and bladder cancer patients using R-Packages(AdequacyModel and zipfR). In first application, we compare BIIIMO-W distribution with Weibull Marshall Olkin-Weibull (WMO-W), odd Burr III Weibull (OBIII-W), Kumaraswamy Weibull (Kum.-W), beta Weibull (Beta-W), generalized gamma Weibull, Weibull and Burr III (BIII) distributions. In second application, we compare BIIIMO-Lindley (BIIIMO-L) distribution with odd Burr III Lindley (OBIII-L), Mc-Donald- Lindley (Mc-L), Kumaraswamy Lindley (Kum.-L), beta Lindley (Beta-L), Weibull power Lindley (W-PL), Kumaraswamy power Lindley (Kum.- PL), Lindley and Burr III (BIII) distributions.

The better fit corresponds to smaller W*, A*, K-S, $-2\hat{\ell}$, AIC, CAIC, BIC and HQIC value. The maximum likelihood estimates (MLEs) of unknown parameters and values of goodness of fit measures are computed for BIIIMO-W, BIIIMO-L distributions and their sub-models and competing models. The MLEs, their standard errors (in parentheses) and goodness-of-fit statistics like W*, A*, K-S (p-value) are given in Tables 4 and 5. Tables 6 and 7 displays goodness-of-fit values.

Data set I

The survival times (days) of 40 patients suffering from leukemia (Abouammoh et al. 1994) are: 115,181, 255, 418, 441, 461, 516, 739, 743,789,807, 865, 924, 983, 1024,1062, 1063,1165, 1191, 1222,1251,1277, 1290,1357,1369, 1408,1455, 1478, 1222,1549, 1578, 1578, 1599, 1603, 1605, 1696, 1735, 1799, 1815,1852.

The BIIIMO-W distribution is best fitted than sub-models and competing models because the values of all criteria are smaller for BIIIMO-W distribution.

We can also perceive that the BIIIMO-W distribution is best fitted model than other sub-models and competing models because BIIIMO-W distribution offers the closer fit to empirical data (Fig. 6).

Data set II

The survival times of 128 bladder cancer patients (Lee and Wang 2003) are 0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79,

Fig. 5 Simulation results of the special BIIIMO-L distribution
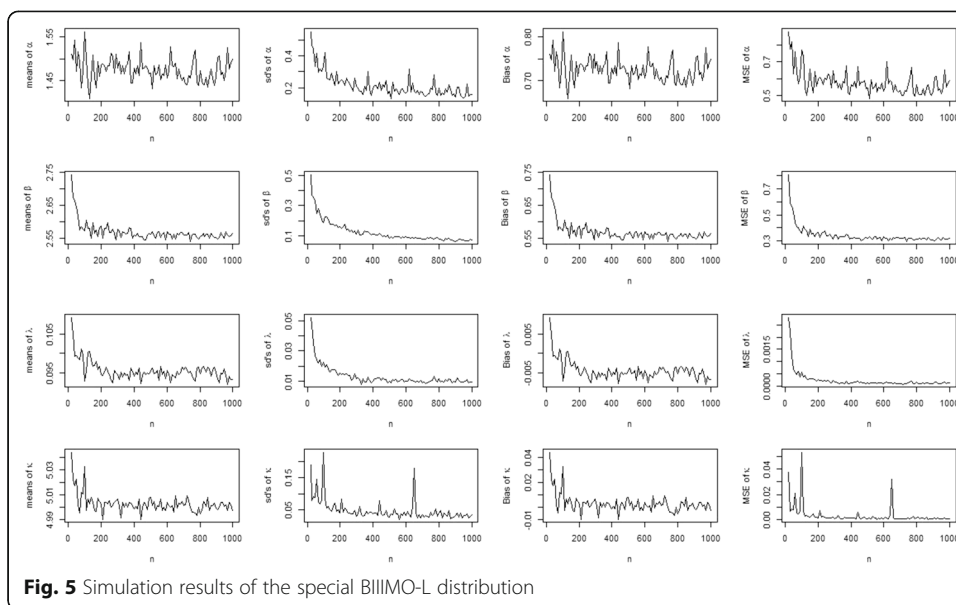
18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69

The BIIIMO-L distribution is best fitted than sub-models and competing models because the values of all criteria are smaller for BIIIMO-L distribution.

We can also perceive that the BIIIMO-L distribution is best fitted model than other sub-models and competing models because BIIIMO-L distribution offers the closer fit to empirical data (Fig. 7).

Table 4 MLEs, their standard errors (in parentheses) and Goodness-of-fit statistics for data set I

Model	α	β	λ	a	b	W^*	A^*	K-S (*p*-value)
BIIIMO-W	0.0778 (0.0535)	61.6872 (50.1271)	15.3084 (8.1416)	–	0.1597 (0.0214)	0.0223	0.1689	0.0657 (0.9952)
WMO-W	–	1.4827 (0.8886)	19.8894 (50.0788)	0.0100 (0.0090)	0.8196 (0.0880)	0.1010	0.6747	0.0917 (0.8897)
OBIII-W	8.3649 (1.9540)	0.3518 (0.2383)	–	0.0071 (0.0027)	0.9864 (0.1077)	0.2146	1.3316	0.1349 (0.4609)
Kum.-W	5.3277 (1.4325)	1.0881 (0.6411)	–	0.0071 (0.0028)	0.8185 (0.0713)	0.4722	2.7197	0.8567 (< 2.2e-16)
Beta-W	686.6493 (275.8812)	117.5352 (76.0463)	–	11.0073 (15.0129)	0.0705 (0.0154)	0.4182	2.4380	0.1769 (0.1637)
GG-W	20.12230 (2.7548)	0.9595 (0.2615)	–	2.4800 (0.1540)	0.3978 (0.1058)	0.3531	2.0916	0.1749 (0.1729)
Weibull	–	–	–	0.0743 (0.3914)	1.9547 (10.3026)	0.4920	2.8243	0.5085 (2.069e-09)
BIII	–	–	–	1000.0297 (545.5985)	1.0548 (0.0879)	0.6829	3.8003	0.3005 (0.001459)

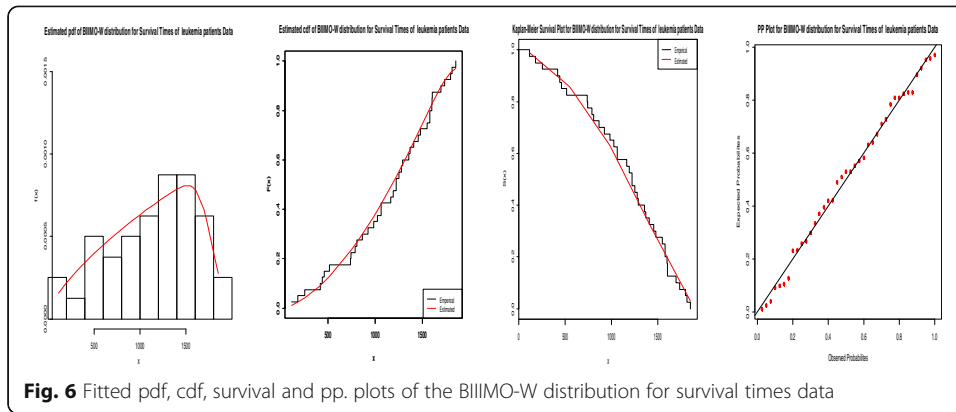**Table 5** MLEs, their standard errors (in parentheses) and Goodness-of-fit statistics for data set II

Model	α	β	λ	a	b	c	W*	A*	K-S (*p*-value)
BIIIMO-L	0.5122 (0.2016)	1.8631 (0.5179)	0.5600 (0.6845)	–	0.1530 (0.1159)	–	0.0168	0.1031	0.0307 (0.9997)
OBIII-L	0.7847 (0.2147)	0.9553 (0.1586)	–	–	0.1723 (0.0330)	–	0.1903	1.1378	0.0934 (0.2145)
Mc-L	1.2106 (0.7307)	0.2846 (0.0286)	–	–	0.4949 (0.0028)	0.7145 (0.5179)	0.1444	0.8629	0.0884 (0.2704)
Kum.-L	0.9611 (0.1671)	0.2825 (0.0276)	–	–	0.5015 (0.0028)	–	0.1344	0.8039	0.0898 (0.2529)
Beta-L	0.9248 (0.1301)	0.2813 (0.0283)	–	–	0.4949 (0.0037)	–	0.1399	0.836	0.0894 (0.2576)
W-PL	0.0215 (0.0285)	13.54214 (12.6100)	–	0.0466 (0.0402)	1.1848 (0.0901)	–	0.1482	0.8851	0.0735 (0.4932)
Kum-PL	3.1406 (2.9576)	1.4481 (3.0649)	–	0.5045 (0.3671)	0.8239 (0.3653)	–	0.0389	0.2541	0.043 (0.9719)
Lindley	–	–	–	–	0.1960 (0.0123)	–	0.1717	1.0257	0.1164 (0.06235)
BIII	4.2070 (0.4054)	1.0333 (0.0604)	–	–	–	–	0.3856	2.4543	0.1017 (0.1413)

Table 6 Goodness-of-fit statistics for data set I

Model	$-2\widehat{\ell}$	AIC	CAIC	BIC	HQIC
BIIIMO-W	598.7174	606.7174	607.8603	613.4729	609.16
WMO-W	606.9106	614.9106	616.0534	621.6661	617.3531
OBIII-W	614.6548	622.6549	623.7977	629.4104	625.0974
Kum.-W	622.6082	630.6082	631.7511	637.3637	633.0508
Beta-W	627.9062	635.9063	637.0491	642.6618	638.3489
GG-W	623.4344	631.4344	632.5772	638.1899	633.8769
Weibull	785.6506	789.6507	789.975	793.0284	790.872
BIII	652.7072	656.7072	657.0316	660.085	657.9285

Table 7 Goodness-of-fit statistics for data set II

Model	$-2\widehat{\ell}$	AIC	CAIC	BIC	HQIC
BIIIMO-L	409.7209	827.4419	827.7671	838.85	832.0771
OBIII-L	416.2479	838.4958	838.6893	847.0519	841.9722
Kum.-L	414.0998	834.1995	834.3931	842.7556	837.6759
Mc-L	414.0916	836.1833	836.5085	847.5914	840.8184
Beta-L	414.1904	834.3807	834.5743	842.9368	837.8571
WPL	414.9072	837.8144	838.1397	849.2226	842.4496
Kum.-PL	410.4152	828.8304	829.1556	840.2386	833.4656
Lindley	419.5299	841.0598	841.0916	843.9118	842.2186
BIII	426.6864	857.3729	857.4689	863.0769	859.6905

Fig. 6 Fitted pdf, cdf, survival and pp. plots of the BIIIMO-W distribution for survival times data
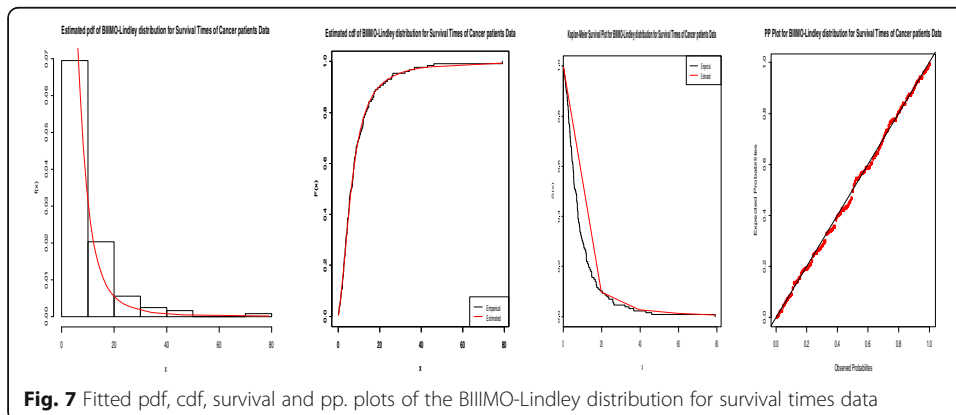
Concluding remarks

We have developed the BIIIMO-G family via the T-X family technique. We have studied properties such as sub-models; descriptive measures based on the quantiles, moments, inequality measures, stress-strength reliability and multicomponent stress-strength reliability model. The BIIIMO-G family is characterized via different techniques. The MLEs for the BIIIMO-G family have been computed. A simulation studies is performed to illustrate the performance of the maximum likelihood estimates (MLEs). Applications of the BIIIMO-G model to real data sets (survival times of leukemia patients and bladder cancer patients data) are presented to show the significance and flexibility of the BIIIMO-G family. Goodness of fit shows that the BIIIMO-G family is a better fit. We have demonstrated that the BIIIMO-G family is empirically better for lifetime applications.

Abbreviations

A*: Anderson-Darling; AIC: Akaike information criterion; Beta- L: Beta Lindley; Beta-W: Beta Weibull; BIC: Bayesian information criterion; BIII: Burr III; BIIIMO-G: Burr III Marshall Olkin-G; BIIIMO-L: Burr III Marshall Olkin-Lindley; BIIIMO-W: Burr III Marshall Olkin-Weibull; CAIC: Consistent Akaike Information Criterion; cdf: Cumulative distribution function; GG-W: Generalized gamma Weibull; GIW-MO: Generalized inverse Weibull-Marshall Olkin; HQIC: Hannan-Quinn information criterion; hrf: Hazard rate function; i.i.d: Independent and identically distributed; KS: Kolmogorov-Smirnov; Kum.-L: Kumaraswamy Lindley; Kum.-PL: Kumaraswamy power Lindley; Kum.-W: Kumaraswamy Weibull; Mc-L: Mc-Donald- Lindley; MLEs: Maximum likelihood estimators; MO-G: Marshall and Olkin; MSE: Mean square error; OBIII-L: Odd Burr III Lindley; OBIII-W: Odd Burr III Weibull; pdf: Probability density function; pp: Probability-Probability; rv: Random variable; sd: Standard deviations; W*: Cramer-von Mises; WMO-W: Weibull Marshall Olkin-Weibull; W-PL: Weibull power Lindley

Fig. 7 Fitted pdf, cdf, survival and pp. plots of the BIIIMO-Lindley distribution for survival times data

Author details
[1]National College of Business Administration and Economics, Lahore, Pakistan. [2]Marquette University, Milwaukee, WI 53201-1881, USA. [3]Department of Measurement and Evaluation, Artvin Çoruh University, Artvin, Turkey. [4]DepartamentodeEstatística, UniversidadeFederaldePernambuco, Recife, PE, Brazil. [5]Department of Statistics Mathematics and Insurance, Benha University, Benha, Egypt.

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